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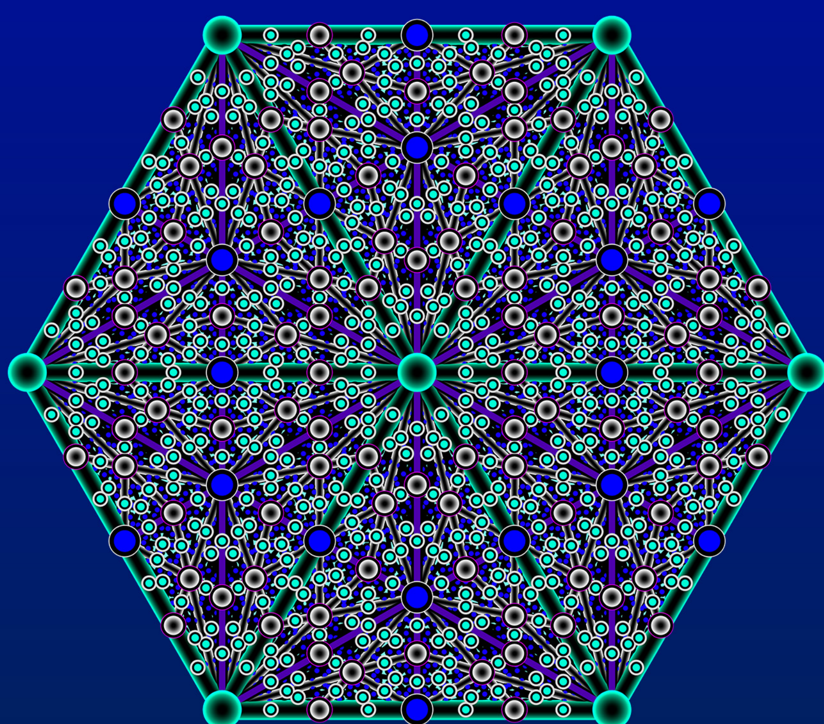

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Geometry of Integrable Linkages

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Abstract: In analogy with the well-known 2-linkage tractor-trailer problem, we define a 2-linkage problem in the plane with novel non-holonomic "no-slip" conditions. Using constructs from sub-Riemannian geometry, we look for geodesics corresponding to linkage motion with these constraints ("tricycle kinematics"). The paths of the three vertices turn out to be critical points for functionals which appear in the hierarchy of conserved quantities for the planar filament equation, a well known completely integrable evolution equation for planar curves. We show that the geodesic equations are completely integrable, and present a second connection to the planar filament equation.

Key words and phrases:
Integrability; Sub-Riemannian Geometry; Elastica

1 Introduction

There has been a good deal of interest recently in what is sometimes informally called "bicycle mathematics". This refers to questions about the geometry of the kinematics of directed line segments in some space (usually R^2 or R^3) subject to a constraint, the "no-slip" condition, which we now describe.

Given a moving oriented line segment of a fixed length, we demand that the velocity of the rear point be parallel to the segment. One imagines the front point to be the front wheel of the bicycle, the second point corresponds to the rear wheel. There are a great number of surprises in the subject, which has a long history and a recent renaissance; for background and history, we refer to [BLPT20], [FLT13], and [Tab17] for more information.

Our focus is on a recent development in the subject involving a fundamental minimization problem: given two oriented line segments S_1, S_2 of equal length, we wish to move S_1 to the position of S_2, where

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the motion is constrained by the no-slip condition indicated above. How can we move S_1 so that the length of the path of the front vertex is minimized? What are the “bicycling geodesics"?

As far as we know, this question was first posed by Maxim Arnold, and addressed in [ABLD+21]. In that paper, the authors use the language and constructs of sub-Riemannian geometry to define a Hamiltonian on the cotangent bundle to the space of segments; the trajectories of the Hamiltonian flow project to arc length-parameterized curves which, at least locally, give the trajectories of shortest distance.

Remarkably, it is shown in [ABLD+21] that these planar curves are *elasticae* - critical curves for the functional $\int_\gamma (\kappa^2+\lambda)\, ds$, where κ is the curvature and the integration is with respect to arc length. We will refer to $\int_\gamma \kappa^2\, ds$ as the *elastic (or bending) energy*.

An analogue of this result exists in R^3 also: if a segment moves through space subject to the no-slip condition, then the trajectory of its front end is a Kirchhoff rod, a critical point of $\int_\gamma (\kappa^2+\mu\tau+\lambda)\, ds$, where τ is the torsion of the space curve, [BJT23]. The bicycling geodesics in Euclidean spaces of higher dimensions are confined to affine 3-dimensional spaces, and the problem reduces to R^3, see [BJT23].

The authors of [ABLD+21] give examples of the mysterious appearance of elastica in many geometric contexts, in addition to the one discovered in their paper. There are more [Per98], and a phenomenon we have observed is that the presence of elastica in a geometrical context often indicates that there is an integrable system floating around in the background.

We now discuss one such context, which will be important to us. Consider the evolution equation on planar curves, the *planar filament flow*, see [LP92],

$$\gamma_t = \frac{\kappa^2}{2} T + \kappa_s N,$$

where (T,N) is the Frenet frame, and κ is the curvature.

This evolution on curves is locally arc-length preserving, so the curve bends without stretching or shrinking. If the curve is closed, then the total length $I_0 = \int_\gamma 1\, ds$ is conserved. In addition, one can check that the elastic energy is also conserved.

In fact, there is an infinite list of such conserved quantities, whose existence can be explained by the close correspondence between the planar filament equation and the modified KdV equation:

$$\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s.$$

If a curve evolves via the planar filament equation, then its curvature evolves via mKdV. The mKdV equation, like its better known “cousin" the KdV equation, is completely integrable, and the planar filament equation inherits this integrability.

Among other things, this means that there is an infinite list of conserved quantities, defined in terms of the curvature and its derivatives. The elastic energy is one of these conserved quantities. We list the first three, as well as their Euler-Lagrange operators:

$$\begin{aligned}
I_0 &= \int_\gamma 1\, ds, \quad E_0 = \kappa(s),\\
I_2 &= \int_\gamma \kappa^2\, ds, \quad E_2 = \kappa'' + \frac{\kappa^3}{2},\\
I_4 &= \int_\gamma (\kappa')^2 - \frac{1}{4}\kappa^4\, ds, \quad E_4 = \frac{5}{2}\kappa^2\kappa'' + \frac{5}{2}\kappa\,\kappa'^2 + \frac{3}{8}\kappa^5 + \kappa''''.
\end{aligned}$$

We will refer to critical points of linear combinations of these functionals as *soliton curves*; for example, a curve satisfying $aE_0 + E_2 = 0$ is a 1-soliton, and a curve satisfying $aE_0 + bE_2 + E_4 = 0$ is a 2-soliton.

With this background in mind, we look at the appearance of elastica in [ABLD$^+$21] in the study of a single line segment, and ask if there might be similar interesting phenomena related to a pair of linked segments.

In analogy with the single segment, we seek to impose appropriate non-holonomic constraints for the motion, construct its associated Hamiltonian, and consider the geodesics on 2-linkage space in the plane, depicted in Figure 1.

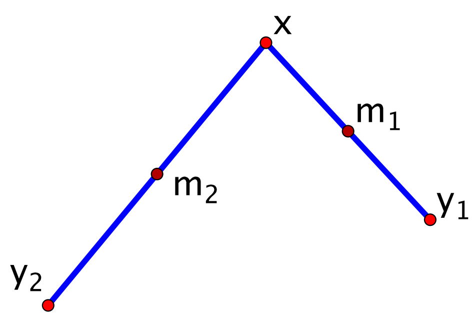

Figure 1: Two-linkage: the no-slip condition is imposed at points m_1 and m_2.

The segments xm_1 and xm_2 have fixed lengths (not necessarily equal). The no slip condition is imposed on points m_1 and m_2, whereas point x can move without restriction. One may think of the linkage xm_1m_2 as a kind of a *tricycle*, where x is the front wheel and m_1 and m_2 are the two back wheels.

The segments xm_1 and xm_2 are extended to xy_1 and xy_2 by doubling their lengths. As we will see, the no slip condition implies that points y_1 and y_2 will move with the same speed as point x and therefore their trajectories have the same lengths as that of point x. The trajectories of points y_1 and y_2 are related to the trajectory of point x by the bicycle (Bäcklund) transformation described in Section 8 below.

The variational problem is to describe the *tricycling geodesics*, the motions of the linkage subject to the above no slip constraint that locally minimize the length of the trajectory of the front point x (and hence those of points y_1 and y_2 as well). In particular, we want to describe the trajectories of the points x, y_1, and y_2, the projections of the respective sub-Riemannian geodesics to the plane.

We now turn to the contents of the paper. In Section 2, following [ABLD$^+$21], we review the equations of 1-linkage geodesic. In Section 3 we show that the distribution in the configuration space of the 2-linkage, described by the no-slip constraint, is completely non-integrable, and in Section 4 we describe the singular curves of this distribution.

In Section 5, we write down the equations of geodesics for 2-linkages and show how 1-linkage geodesics "lift" to 2-linkage geodesics. Next, in Section 6, we study the problem of 2-linkages when the segments have equal length. Surprisingly, it turns out that the path of the "front" point x of the 2-linkage is a 1-soliton (elastica). Unlike the case of 1-linkages, we obtain both non-inflectional and inflectional elasticae.

Section 7 presents a different, computer-assisted, method of obtaining the results of Section 6, using Gröbner bases. We use this method later in the paper, when the calculations become too overwhelming.

Section 8 is an aside on using Bäcklund transformations (defined at that point) to take 1-soliton curves to 2-soliton curves. Using these results, we conclude that (in the equal lengths case) the paths associated with the other two end points are 2-soliton curves.

Section 9 presents a variety of examples when the path of the “front" point x is an inflectional elastica. We also show the trajectories of points y_1 and y_2 and depict the motion of the 2-linkage.

Section 10 deals with the case of unequal lengths; in this case, all three of the paths swept out by the three vertices x, y_1, y_2 are 2-solitons. Their curvatures satisfy $aE_0 + bE_2 + E_4 = 0$, with the same parameters a, b for all three points (this does not imply that the three paths are congruent, although that can happen). The constants a and b are expressible in terms of the phase space variables. Both are constants of motion for our Hamiltonian flow on the linkage space, with b functionally dependent upon a.

In Section 11, we discuss complete integrability of our Hamiltonian system in the equal lengths case. In Section 12, we give a geometric interpretation of the flow of the “fourth" integral (the above mentioned expression b), a Hamiltonian in its own right, and give a geometric interpretation of its flow in terms of the planar filament equation.

We conclude with Section 13 that presents a variety of open problems extending, and motivated by, the present research. We illustrate the complexity of some of these problems with computer graphics.

This investigation of the geometry of linkages has involved numerical, graphical, and computer algebra experimentation. For example, Proposition 4.2 and Theorem 4 were both conjectured after graphing solutions to the relevant differential equations, followed by rigorous proof.

In parts of the paper, we lean heavily on computer algebra tools, rather than explicit “hand” calculations, for proofs. Yet, we want the reader to be able to check the details of those calculations. That is why we provide enough information, so that someone with moderate facility with any one of the standard computer algebra systems can reproduce our results. We look forward to the discovery of proofs which do not require the “brute force" calculations implicit in these computer algebra computations.

2 Review of 1-linkage geometry

In this section we briefly review paper [ABLD+21] that, along with [BJT23], has motivated the present study.

Consider the space of oriented segments of a fixed (say, unit) length in the plane. The segment can move in such a way that the velocity of its rear end is aligned with the segment. As we mentioned earlier, this is a model of the bicycle kinematics: the rear wheel of the bicycle is fixed on its frame.

The configuration space $\mathcal{C}$ of the segments is 3-dimensional, and the above described non-holonomic constraint defines a 2-dimensional distribution $\mathcal{D}$ therein. This distribution is completely non-integrable, and $\mathcal{C}$ is the space of oriented contact elements in R^2 with its standard contact structure. Due to the Chow-Rashevskii theorem, every pair of points can be connected by a horizontal curve, that is, a curve tangent to $\mathcal{D}$ (see, e.g., [Mon06]).

The kernel of the differential of the projection $\mathcal{C} \to R^2$ on the front end of the segment is transverse to $\mathcal{D}$, and this differential maps the planes of the distribution isomorphically onto R^2. One gives $\mathcal{D}$ a Riemannian metric by pulling back the Euclidean metric from the plane. The problem is to describe the geodesics of this sub-Riemannian metric. We call such geodesics minimizing bicycle paths or bicycling geodesics. Let us reiterate: the length of the bicycle path by definition is the length of the front track.

One uses Hamiltonian formalism to study this problem. Choose an orthonormal frame (v_1, v_2) of $\mathcal{D}$, and consider the vector fields v_1 and v_2 as linear functions L_1 and L_2 on the cotangent bundle $T^*\mathcal{C}$. One obtains a Hamiltonian $H = \frac{1}{2}(L_1^2 + L_2^2)$, and the projections to $\mathcal{C}$ of the trajectories of the respective Hamiltonian vector field on the fixed energy surface $H = \frac{1}{2}$ are the arc length parameterized sub-Riemannian geodesics. See [Mon06] for details of this theory.

The main results of [ABLD$^+$21] are as follows.

Theorem 1. *(i) The front track of a minimizing bicycle path is a straight line or an arc of a non-inflectional elastic curve. Every shape of non-inflectional elasticae arise in this way.*
(ii) An infinitely long bicycle path is a global minimizer (all its segments minimize length between their end points) if either the front track is a straight line and the rear track is either a straight line or a tractrix, or the front track is an Euler soliton and the rear track is a tractrix.

We recall that elastica are the curves that have critical bending energy among the curves of fixed length connecting two fixed points; see, e.g., [Sin08]. See Figure 2, borrowed from [ABLD$^+$21], for various shapes of elastic curves.

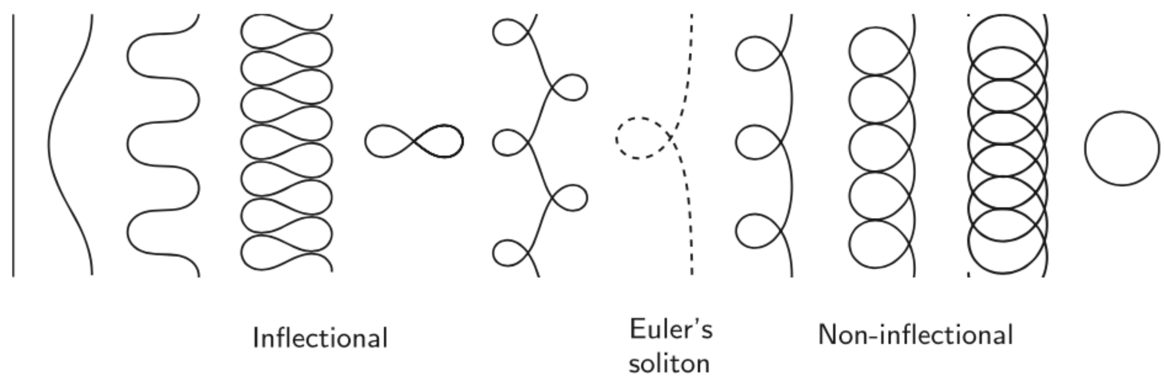

Figure 2: A variety of elastic curves.

In the bicycle terminology, a tractrix is the rear track when the front track is a straight line. The Euler soliton is obtained from the straight line by what is known as the bicycle (or Bäcklund) transformation, consisting of rotation of the bicycle through 180° about its rear end; this transformation is an isometry of $\mathcal{C}$.

3 Non-integrability of the distribution

We start with a description of the configuration space of the linkage and of the distribution defined by the no slip condition.

Introduce the following coordinates: $x = (x_1, x_2)$, the Cartesian coordinates of point x, and α_1, α_2, the directions of the oriented segments m_1x and m_2x, respectively. The lengths of the segments m_1x and m_2x

are denoted by ℓ_1 and ℓ_2. The configuration space of the 2-linkage is $R^2 \times T^2$; let $\mathcal{D}$ be the 2-dimensional distribution defined by the non-holonomic constraint.

We will further assume that $m_1 \neq m_2$, that is, if $\ell_1 = \ell_2$, then $\alpha_1 \neq \alpha_2$. Denote this reduced configuration space by $\mathcal{C}$. As in the 1-linkage case, the kernel of the differential of the projection $\mathcal{C} \to R^2$ is transverse to $\mathcal{D}$, and we give the distribution the pullback metric.

Theorem 2. *The distribution $\mathcal{D}$ in $\mathcal{C}$ is completely non-integrable with the growth vector $(2,3,4)$.*

Proof. One has

$$m_1 = (x_1 - \ell_1 \cos\alpha_1, x_2 - \ell_1 \sin\alpha_1), \ \ m_2 = (x_2 - \ell_2 \cos\alpha_2, x_2 - \ell_2 \sin\alpha_2),$$

the distribution is given by the Pfaffian equations

$$\lambda_1 = \sin\alpha_1\ dx_1 - \cos\alpha_1\ dx_2 + \ell_1 d\alpha_1 = 0, \ \ \lambda_2 = \sin\alpha_2\ dx_1 - \cos\alpha_2\ dx_2 + \ell_2 d\alpha_2 = 0,$$

and a basis of horizontal fields is given by

$$v_1 = \partial_{x_1} - \frac{\sin\alpha_1}{\ell_1}\,\partial_{\alpha_1} - \frac{\sin\alpha_2}{\ell_2}\,\partial_{\alpha_2}, \ \ v_2 = \partial_{x_2} + \frac{\cos\alpha_1}{\ell_1}\,\partial_{\alpha_1} + \frac{\cos\alpha_2}{\ell_2}\,\partial_{\alpha_2}. \tag{1}$$

Then

$$[v_1, v_2] = \frac{1}{\ell_1^2}\,\partial_{\alpha_1} + \frac{1}{\ell_2^2}\,\partial_{\alpha_2}, \ \ [[v_1,v_2],v_1] = -\frac{\cos\alpha_1}{\ell_1^3}\,\partial_{\alpha_1} - \frac{\cos\alpha_2}{\ell_2^3}\,\partial_{\alpha_2},$$
$$[[v_1,v_2],v_2] = -\frac{\sin\alpha_1}{\ell_1^3}\,\partial_{\alpha_1} - \frac{\sin\alpha_2}{\ell_2^3}\,\partial_{\alpha_2}.$$

If $\ell_1 \neq \ell_2$, then the matrix

$$\begin{pmatrix} \frac{1}{\ell_1^2} & \frac{\cos\alpha_1}{\ell_1^3} & \frac{\sin\alpha_1}{\ell_1^3} \\ \frac{1}{\ell_2^2} & \frac{\cos\alpha_2}{\ell_2^3} & \frac{\sin\alpha_2}{\ell_2^3} \end{pmatrix} \tag{2}$$

has rank 2. Indeed, if

$$\left(\frac{\cos\alpha_1}{\ell_1^3}, \frac{\cos\alpha_2}{\ell_2^3}\right) = t\left(\frac{1}{\ell_1^2}, \frac{1}{\ell_2^2}\right), \ \ \left(\frac{\sin\alpha_1}{\ell_1^3}, \frac{\sin\alpha_2}{\ell_2^3}\right) = s\left(\frac{1}{\ell_1^2}, \frac{1}{\ell_2^2}\right),$$

then $(t^2+s^2)\ell_1^2 = (t^2+s^2)\ell_2^2$, and hence $\ell_1 = \ell_2$.

Since the matrix has rank 2, both ∂_{α_1} and ∂_{α_2} are linear combinations of $[v_1,v_2], [[v_1,v_2],v_1]$ and $[[v_1,v_2],v_2]$, and then both ∂_{x_1} and ∂_{x_2} are linear combinations of these vectors, and v_1 and v_2.

If $\ell_1 = \ell_2$, we may assume that they are equal to 1. In this case the rank of matrix (2) is not equal to 2 only if $\alpha_1 = \alpha_2$, which is excluded. In addition,

$$v_2 + [[v_1,v_2],v_1] = \partial_{x_2}, \ \ v_1 - [[v_1,v_2],v_2] = \partial_{x_2},$$

as needed. □

The Chow-Rashevskii theorem implies that any pair of points of $\mathcal{C}$ can be connected by a horizontal curve.

A 2-dimensional distribution on a 4-dimensional manifold with the growth vector $(2,3,4)$ is called an Engel structure. Like contact structures, all Engel structures are locally diffeomorphic. The normal form of the distribution is given, in local coordinates (x,y,z,w), by the two 1-forms $dz - ydx$ and $dy - wdx$, see [Mon06].

4 Singular curves

A specific phenomenon of sub-Riemannian geometry is that the space of horizontal paths that connect two fixed points may have singularities. These singular curves can be sub-Riemannian geodesics that are not detected by the Hamiltonian formalism of Section 5; see [Mon06, Mon95].

It is known, see [BH93], that Englel manifolds are foliated by singular curves: in the above mentioned normal form, they are the trajectories of the vector field $\partial/\partial w$. Let us describe the singular curves in our setting.

Theorem 3. *The singular curves in $\mathcal{C}$ are the integral curves of the vector field*

$$\xi = \left(\frac{\sin\alpha_1}{\ell_1} - \frac{\sin\alpha_2}{\ell_2}\right) v_1 - \left(\frac{\cos\alpha_1}{\ell_1} - \frac{\cos\alpha_2}{\ell_2}\right) v_2,$$

where v_1 and v_2 are as in (1).

Proof. Set $\mathcal{D}^2 = \mathcal{D} + [\mathcal{D},\mathcal{D}]$. A desired vector field $\xi \in \mathcal{D}$ is characterized by the property that $[\xi,\mathcal{D}^2] \subset \mathcal{D}^2$, see [Mon95].

The differential form $\lambda := \ell_1\lambda_1 - \ell_2\lambda_2$ annihilates $\mathcal{D}^2$. Let $\xi = fv_1 + gv_2$. Then the fields $[\xi, v_1]$ and $[\xi, v_2]$ are already in ker λ. One has

$$\lambda\left(\left[\frac{1}{\ell_1^2}\,\partial_{\alpha_1} + \frac{1}{\ell_2^2}\,\partial_{\alpha_2}, \xi\right]\right) = f\left(\frac{\cos\alpha_1}{\ell_1} - \frac{\cos\alpha_2}{\ell_2}\right) + g\left(\frac{\sin\alpha_1}{\ell_1} - \frac{\sin\alpha_2}{\ell_2}\right).$$

This must vanish, hence one can choose

$$f = \frac{\sin\alpha_1}{\ell_1} - \frac{\sin\alpha_2}{\ell_2},\ \ g = -\frac{\cos\alpha_1}{\ell_1} + \frac{\cos\alpha_2}{\ell_2},$$

as claimed. □

The explicit formula is as follows:

$$\begin{aligned}\xi = &\left(\frac{\sin\alpha_1}{\ell_1} - \frac{\sin\alpha_2}{\ell_2}\right)\,\partial_{x_1} - \left(\frac{\cos\alpha_1}{\ell_1} - \frac{\cos\alpha_2}{\ell_2}\right)\,\partial_{x_2} \\ &- \left(\frac{1}{\ell_1^2} - \frac{\cos(\alpha_1-\alpha_2)}{\ell_1\ell_2}\right)\,\partial_{\alpha_1} + \left(\frac{1}{\ell_2^2} - \frac{\cos(\alpha_1-\alpha_2)}{\ell_1\ell_2}\right)\,\partial_{\alpha_2}.\end{aligned}$$

Proposition 4.1. *If $\ell_1 = \ell_2$, then the motion along the singular curves is as follows: the angle $(\alpha_1+\alpha_2)/2$ remains constant, and point x moves along a straight line that bisects the angle between the segments xm_1 and xm_2. The points m_1 and m_2 move along the tractrices which are symmetric with respect to this line. This singular curve is a minimizing geodesic.*

Proof. If $\ell_1 = \ell_2 = 1$, the formula simplifies to

$$\begin{aligned}&\xi = 2\sin\left(\frac{\alpha_1-\alpha_2}{2}\right)\times \\ &\left[\cos\left(\frac{\alpha_1+\alpha_2}{2}\right)\partial_{x_1} + \sin\left(\frac{\alpha_1+\alpha_2}{2}\right)\partial_{x_2} + \sin\left(\frac{\alpha_1-\alpha_2}{2}\right)(\partial_{\alpha_2} - \partial_{\alpha_1})\right].\end{aligned}$$

The factor in front does not vanish on $\mathcal{C}$, and we arrive at the differential equations describing the singular curves:

$$\dot{x}_1 = \cos\left(\frac{\alpha_1+\alpha_2}{2}\right), \;\; \dot{x}_2 = \sin\left(\frac{\alpha_1+\alpha_2}{2}\right),$$
$$\dot{\alpha}_1 = -\sin\left(\frac{\alpha_1-\alpha_2}{2}\right), \;\; \dot{\alpha}_2 = \sin\left(\frac{\alpha_1-\alpha_2}{2}\right).$$

It follows that $\dot{\alpha}_1+\dot{\alpha}_2 = 0$, hence $\alpha_1+\alpha_2$ is constant, and then

$$\dot{x} = \left(\cos\left(\frac{\alpha_1+\alpha_2}{2}\right), \sin\left(\frac{\alpha_1+\alpha_2}{2}\right)\right),$$

as claimed.

The last claim follows from the fact the the trajectory of point x is a straight line. $\square$

We also have the following

Proposition 4.2. *If $\ell_1 \neq \ell_2$, then the x-projection of a singular curve on the plane is an elastic curve.*

Proof. We reiterate the differential equations for the singular geodesic:

$$\frac{d}{dt}\alpha_1(t) = \frac{\ell_1\cos(-\alpha_2(t)+\alpha_1(t))-\ell_2}{\ell_1^2\ell_2}$$
$$\frac{d}{dt}\alpha_2(t) = -\frac{\ell_2\cos(-\alpha_2(t)+\alpha_1(t))-\ell_1}{\ell_1\ell_2^2}$$
$$\frac{d}{dt}x_1(t) = \frac{\sin(\alpha_1(t))\,\ell_2-\sin(\alpha_2(t))\,\ell_1}{\ell_1\ell_2}$$
$$\frac{d}{dt}x_2(t) = -\frac{\cos(\alpha_1(t))\,\ell_2-\cos(\alpha_2(t))\,\ell_1}{\ell_1\ell_2}$$

We experiment, and plot the planar projection of the solution curve (the following observations are independent of the particular choice of initial conditions), see Figure 3.

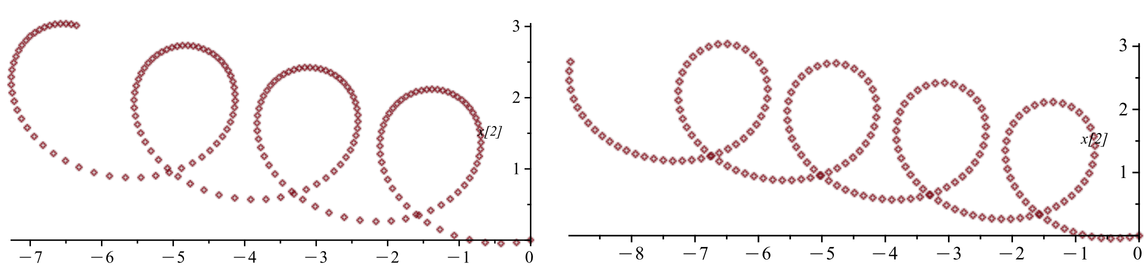

Figure 3: Left: planar projection of a singular geodesic; right: the same curve parameterized by arc length.

The curve looks like an elastica and, by plotting points equally spaced in the parameter t, we see that it is not unit speed parameterized.

We will fix this, and then compute the curvature of the planar curve. Let S be the speed; it is easily computed to be:

$$S = \sqrt{-\frac{2\ell_1\ell_2\cos(-\alpha_2(t)+\alpha_1(t)) - \ell_1^2 - \ell_2^2}{\ell_1^2\ell_2^2}}.$$

Dividing our set of equations by S, we obtain a time-scaled version of the differential equations, with the same solution curves (reparameterized). We plot solutions to these modified equations, and obtain the same planar curve, but unit speed parameterized, see Figure 3.

Let $\mathbf{p} = (x_1(t), x_2(t))$, the curve defined by the second pair of differential equations. By scaling, we have $\dot{p} = T$, the unit tangent to the curve. One immediately obtains the unit normal N, and the computation $< \dot{T}, N >$ results in the curvature. We differentiate the curvature twice:

$$\begin{aligned}
\kappa &= \frac{\ell_1^2 - \ell_2^2}{\ell_1\ell_2\sqrt{-2\ell_1\ell_2\cos(-\alpha_2+\alpha_1) + \ell_1^2 + \ell_2^2}},\\
\dot{\kappa} &= \frac{\left(-\ell_1^2+\ell_2^2\right)\sin(-\alpha_2+\alpha_1)}{\ell_1\ell_2\left(2\ell_1\ell_2\cos(-\alpha_2+\alpha_1) - \ell_1^2 - \ell_2^2\right)}.\\
\ddot{\kappa} &= \frac{\cos(-\alpha_2+\alpha_1)\,\ell_1^4 - \cos(-\alpha_2+\alpha_1)\,\ell_2^4 - 2\ell_1^3\ell_2 + 2\ell_1\ell_2^3}{\ell_1^2\ell_2^2\left(2\ell_1\ell_2\cos(-\alpha_2+\alpha_1) - \ell_1^2 - \ell_2^2\right)\sqrt{-2\ell_1\ell_2\cos(-\alpha_2+\alpha_1) + \ell_1^2 + \ell_2^2}},
\end{aligned}$$

and compute the expression $\ddot{\kappa} + \frac{\kappa^3}{2} + A\kappa$. It vanishes if $A = -\frac{\ell_1^2+\ell_2^2}{2\ell_1^2\ell_2^2}$, hence the curve is an elastica. $\square$

5 Geodesic equations for 2-linkages

Now we apply the Hamiltonian formalism to deduce the equations for non-singular geodesics.

An orthonormal basis of horizontal vector fields is given by (v_1, v_2) from equation (1). Consider the following coordinates in $T^*\mathcal{C}$:

$$(x_1, x_2, \alpha_1, \alpha_2, p_1, p_2, \eta_1, \eta_2),$$

where the last four are the fiber coordinates conjugated to the first four, respectively, that is, the canonical symplectic structure in $T^*\mathcal{C}$ is

$$dx_1 \wedge dp_1 + dx_2 \wedge dp_2 + d\alpha_1 \wedge d\eta_1 + d\alpha_2 \wedge d\eta_2.$$

Then the vector fields v_1 and v_2 become linear functions on $T^*\mathcal{C}$:

$$L_1 = p_1 - \frac{1}{\ell_1}\eta_1\sin\alpha_1 - \frac{1}{\ell_2}\eta_2\sin\alpha_2,\ \ L_2 = p_2 + \frac{1}{\ell_1}\eta_1\cos\alpha_1 + \frac{1}{\ell_2}\eta_2\cos\alpha_2. \tag{3}$$

The Hamiltonian is given by

$$H = \frac{1}{2}(L_1^2 + L_2^2).$$

Set $H = \frac{1}{2}$, and introduce the angle γ by setting $L_1 = \cos\gamma, L_2 = \sin\gamma$.

The Hamiltonian equations read (after an application of appropriate trigonometric identities):

$$\begin{aligned} &\dot{x}_1 = \cos\gamma,\ \ \dot{x}_2 = \sin\gamma,\ \ \dot{\alpha}_1 = \frac{\sin(\gamma-\alpha_1)}{\ell_1},\ \ \dot{\alpha}_2 = \frac{\sin(\gamma-\alpha_2)}{\ell_2},\\ &\dot{p}_1 = \dot{p}_2 = 0,\ \ \dot{\eta}_1 = \frac{\eta_1\cos(\gamma-\alpha_1)}{\ell_1},\ \ \dot{\eta}_2 = \frac{\eta_2\cos(\gamma-\alpha_2)}{\ell_2}, \end{aligned} \tag{4}$$

where dot is the time derivative.

We are interested in the curve $x(t)$. Note that t is the arc length parameter; let $\kappa(t)$ be the curvature of this curve.

Proposition 5.1. *If $x(t)$ satisfies equations (4) then*

$$\begin{aligned} &\kappa = \frac{\eta_1}{\ell_1^2} + \frac{\eta_2}{\ell_2^2},\ \ \dot{\kappa} = \frac{\eta_1\cos(\gamma-\alpha_1)}{\ell_1^3} + \frac{\eta_2\cos(\gamma-\alpha_2)}{\ell_2^3},\\ &\ddot{\kappa} = \frac{\eta_1}{\ell_1^4} + \frac{\eta_2}{\ell_2^4} - \kappa\left[\frac{\eta_1\sin(\gamma-\alpha_1)}{\ell_1^3} + \frac{\eta_2\sin(\gamma-\alpha_2)}{\ell_2^3}\right]. \end{aligned} \tag{5}$$

Proof. One has $\kappa = \dot{\gamma}$. Differentiate (say) the second equation in (3), using that $\dot{p}_2 = 0$. Use equations (4) and some trigonometry to obtain

$$\dot{\gamma}\sin\gamma = \frac{\eta_1}{\ell_1^2}\sin\gamma + \frac{\eta_2}{\ell_2^2}\sin\gamma,$$

which implies the equation for the curvature. The other two equations are obtained by differentiating the first one, substituting the values of the derivatives from (4), and using suitable trigonometric identities. □

Remark 5.2. The equations

$$\dot{\alpha}_1 = \frac{\sin(\gamma-\alpha_1)}{\ell_1},\ \ \dot{\alpha}_2 = \frac{\sin(\gamma-\alpha_2)}{\ell_2}$$

in (4) are just the "bicycle equations" expressing the no-skid non-holonomic constraints on the points m_1 and m_2, see, e.g., [BLPT20, FLT13].

Lifting 1-linkage geodesics to 2-linkage geodesics Consider 1-linkage xm_1, and let $x(t)m_1(t), t \in [0,T]$, be a bicycling geodesic. Let $m_2 = m_2(0)$ be another point, interpreted as the rear wheel of the bicycle xm_2. The no-skid constraint uniquely determines the motion $m_2(t), t \in [0,T]$, and we obtain a horizontal path in the 2-linkage configuration space $\mathcal{C}$.

Proposition 5.3. *This path is geodesic.*

Proof. We need to show that the path $x(t)m_1(t)m_2(t)$ minimizes the distance between its two sufficiently close points.

Assume that there exists a shorter horizontal path from $x(t_1)m_1(t_1)m_2(t_1)$ to $x(t_2)m_1(t_2)m_2(t_2)$. Forgetting point m_2, we obtain a shorter horizontal path in the space of 1-linkage connecting $x(t_1)m_1(t_1)$ and $x(t_2)m_1(t_2)$. This contradicts the assumption that $x(t)m_1(t)$ is a bicycling geodesic. □

If $x(t)m_1(t)$ is a bicycling geodesic, then the curve $x(t)$ is a non-inflectional elastica, see Section 2. The above proposition implies that all non-inflectional elasticae are the projections of some geodesics in $\mathcal{C}$ to R^2. It also follows that there are infinitely many lifts of a non-inflectional elastica to a geodesic curve in $\mathcal{C}$.

6 The case of equal length line segments and the appearance of elastica

In this section we assume that $\ell_1 = \ell_2$ and, by scaling, that this length is unit.

Theorem 4. *The projection $x(t)$ of a geodesic in $\mathcal{C}$ to R^2 is an elastic curve.*

Proof. Elasticae are characterized by the differential equation on their curvature as a function of the arc length parameter

$$\ddot{\kappa} + \frac{1}{2}\kappa^3 + A\kappa = 0,$$

so (ignoring the case of a straight line where the curvature is identically zero) we need to show that

$$\frac{\ddot{\kappa}}{\kappa} + \frac{1}{2}\kappa^2 = const.$$

Equation (5) implies that

$$\frac{\ddot{\kappa}}{\kappa} = 1 - \eta_1 \sin(\gamma - \alpha_1) - \eta_2 \sin(\gamma - \eta_2).$$

One can solve equations (3) for η_1 and η_2

$$\eta_1 = \frac{\cos(\gamma - \alpha_2) - p_1 \cos\alpha_2 - p_2 \sin\alpha_2}{\sin(\alpha_2 - \alpha_1)}, \quad \eta_2 = \frac{\cos(\gamma - \alpha_1) - p_1 \cos\alpha_1 - p_2 \sin\alpha_1}{\sin(\alpha_1 - \alpha_2)} \tag{6}$$

and substitute to the above equation to obtain, using trigonometric identities:

$$\frac{\ddot{\kappa}}{\kappa} = p_1 \cos\gamma + p_2 \sin\gamma. \tag{7}$$

We want to check that

$$\frac{d}{dt}\left(\frac{\ddot{\kappa}}{\kappa}\right) + \kappa\dot{\kappa} = 0.$$

One has

$$\frac{d}{dt}(p_1 \cos\gamma + p_2 \sin\gamma) = (-p_1 \sin\gamma + p_2 \cos\gamma)\kappa,$$

so it remains to check that

$$-p_1 \sin\gamma + p_2 \cos\gamma = -\dot{\kappa}. \tag{8}$$

To see this, take the value of $\dot{\kappa}$ from (5) and substitute η_1 and η_2 from (6) to obtain a true identity. This shows that $x(t)$ is an elastic curve. $\square$

Let us write the equations of elastica in two forms:

$$\frac{\ddot{\kappa}}{\kappa}+\frac{1}{2}\kappa^2+A=0,\quad \dot{\kappa}^2+\frac{1}{4}\kappa^4+A\kappa^2+B=0, \tag{9}$$

the second one is obtained from the first by integration.

Lemma 6.1. *One has $A^2-B=p_1^2+p_2^2$.*

Proof. Equations (7) and (8) imply

$$\left(\frac{\ddot{\kappa}}{\kappa}\right)^2+\dot{\kappa}^2=p_1^2+p_2^2.$$

Hence

$$\left(\frac{1}{2}\kappa^2+A\right)^2-\frac{1}{4}\kappa^4-A\kappa^2-B=p_1^2+p_2^2,$$

therefore

$$A^2-B=p_1^2+p_2^2,$$

as claimed. □

Set

$$G=-A=\frac{\ddot{\kappa}}{\kappa}+\frac{1}{2}\kappa^2.$$

As we have seen, this is another integral of motion, in addition to H, p_1, and p_2. We shall address the independence of these integrals in Section 11.

Using (7), we also obtain a differential equation on γ:

$$\dot{\gamma}=\kappa=\sqrt{2(G-p_1\cos\gamma-p_2\sin\gamma)}.$$

The substitution $y=\tan(\gamma/2)$ transforms this to the equation $\dot{y}=\sqrt{Q(y)}$, where Q is a polynomial of degree 4.

Note that all non-closed elastic curves are contained in a strip, see Figure 2. Let us call the direction of the strip the direction of the respective elastica.

Lemma 6.2. *The vector (p_1,p_2) is parallel to the direction of the respective elastica.*

Proof. Write $(p_1,p_2)=r(\cos\phi,\sin\phi)$, so ϕ is the direction of this vector. Then equation (8) can be written as $\dot{\kappa}=r\sin(\gamma-\phi)$. Hence if $\dot{\kappa}=0$, then $\phi=\gamma \mod \pi$. That is, ϕ is the direction of the elastica at its vertices (the point where the curvature is extremal). This direction coincides with the direction of the elastica, see [Sin08] and Figure 2. □

This lemma agrees with the fact that when the elastica is a circle (all points are vertices), one has $p_1=p_2=0$. We also note that $r=|(p_1,p_2)|$ is determined by the function $\gamma(t)$:

$$r=\left|\frac{\dot{\kappa}}{\sin(\gamma-\phi)}\right|=\left|\frac{\ddot{\gamma}}{\sin(\gamma-\phi)}\right|.$$

One can reduce the number of variables in equations (4) by solving equations (3) for η_1 and η_2, and substituting the result in (4). This yields the system of equations

$$\begin{aligned}
&\dot{\alpha}_1 = \sin(\gamma-\alpha_1),\ \dot{\alpha}_2 = \sin(\gamma-\alpha_2),\\
&\dot{\gamma} = \frac{\sin\left(\gamma-\frac{\alpha_1+\alpha_2}{2}\right)+p_1\sin\left(\frac{\alpha_1+\alpha_2}{2}\right)-p_2\cos\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\cos\left(\frac{\alpha_2-\alpha_1}{2}\right)}.
\end{aligned} \tag{10}$$

As we saw in Section 5, and in view of Section 2, every non-inflectional elastica is a projection of a geodesic curve. What about inflectional elasticae?

Theorem 5. *Every shape of elastica appears as the projection of a geodesic curve in $\mathcal{C}$.*

Proof. Given an arc length parameterized elastica, we have an explicit function $\gamma(t)$ (given in terms of elliptic functions). Then the constants p_1 and p_2 are determined – see Lemma 6.2 and the discussion after it.

Consider equations (10). Once the initial values $\alpha_1(0)$ and $\alpha_2(0)$ are chosen, the first two differential equations determine the functions $\alpha_1(t)$ and $\alpha_2(t)$ uniquely.

Suppose that $\alpha_1(0)$ and $\alpha_2(0)$ are chosen in such a way that the third equation (10) holds for $t=0$. We claim that then it holds for all values of t. This would imply that the elastica under consideration is the projection of the respective geodesic.

Set

$$f(t) = \dot{\gamma} - \frac{\sin\left(\gamma-\frac{\alpha_1+\alpha_2}{2}\right)+p_1\sin\left(\frac{\alpha_1+\alpha_2}{2}\right)-p_2\cos\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\cos\left(\frac{\alpha_2-\alpha_1}{2}\right)}.$$

Using the differential equations for $\alpha_1(t)$ and $\alpha_2(t)$, and equation (8), one calculates that f satisfies the following first order linear differential equation

$$\dot{f} = -\frac{\cos\left(\gamma-\frac{\alpha_1+\alpha_2}{2}\right)}{\cos\left(\frac{\alpha_1-\alpha_2}{2}\right)} f.$$

Since $f(0)=0$ by assumption, $f(t)\equiv 0$ is a solution. Then the standard theorem of existence and uniqueness of solutions to ordinary differential equations (e.g.,[CL55]) implies that the third equation (10) holds for all t, as needed. □

The shape of the elastica satisfying equation (9) is determined by the scale-invariant parameter $\mu = B/A^2$ (our choice of constant B is slightly different from that in [ABLD+21]). One has

$$\mu = 1 - \frac{p_1^2+p_2^2}{G} \leq 1.$$

A non-inflectional elastica corresponds to $\mu > 0$, and an inflectional elastica to $\mu < 0$.

Thus, our situation differs from that occuring in [ABLD+21]; for the equal lengths case of 2-linkages, μ may be negative; this fact allows for inflectional elasticae. In Example 9.2, we show a convenient way to generate all of the inflectional elasticae.

7 Computer-assisted proofs

It is perhaps not surprising that calculations for 2-linkages with unequal length line segments are harder than in the equal length case. When we have been unable to prove results “by hand”, we have resorted to computer algebra calculations.

In particular, we make elementary but extensive use of a particular tool in computer algebra, the use of *Gröbner* bases, which allows us to use “brute force" to do the relevant calculations. Gröbner basis tools exist for all of the major computer algebra systems (in particular, the popular tools, Maple and Mathematica). The paper [Stu05] gives a nice short introduction, but we say a few words here.

Given an ideal I in a polynomial ring $R = F[x_1, x_2, \dots, x_n]$ with generating polynomials $PB = \{p_1, p_2, \dots, p_k\}$, how can one decide in an algorithmic manner whether another polynomial p is an element of the ideal? And, if p is not an element of I, is there a canonical representative of p in the quotient ring $R \mod I$? This question was addressed by Buchberger; the algorithm requires the computation of an alternative set of generators for I (the *Gröbner basis PGB*), and then a computation using this new basis to find the canonical representative of $p \mod I$. See, e.g., [Laz83].

Of course, in order for this to be useful, one must phrase the problem of interest entirely in terms of polynomial calculations. We show how this applies to a second proof that, for equal lengths, the curve $x(t)$ is an elastic curve. We describe this computation in excruciating detail, and we will be more terse later in the paper.

The idea is to express all quantities and equations as polynomial expressions in the set of variables

$$vars = \{\alpha_1, \alpha_2, \eta_1, \eta_2, p_1, p_2, \mathbf{c}\alpha_1, \mathbf{c}\alpha_2, \mathbf{s}\alpha_1, \mathbf{s}\alpha_2\},$$

where $\mathbf{c}\alpha_1, \mathbf{c}\alpha_2, \mathbf{s}\alpha_1, \mathbf{s}\alpha_2$ are polynomial variables corresponding to the functions $\cos\alpha_1, \cos\alpha_2, \sin\alpha_1, \sin\alpha_2$ respectively. For example, the polynomial representing the Hamiltonian can be written as

$$\begin{aligned} H \longleftrightarrow Hpoly = &-\frac{\mathbf{s}\alpha_1\,\eta_1 p_1}{\ell_1} - \frac{\mathbf{s}\alpha_2\,\eta_2 p_1}{\ell_2} + \frac{\eta_1\,\mathbf{c}\alpha_1\,p_2}{\ell_1} + \frac{\eta_2\,\mathbf{c}\alpha_2\,p_2}{\ell_2} + \frac{p_1^2}{2} + \frac{p_2^2}{2} \\ &+ \frac{\eta_1\,\mathbf{c}\alpha_1\,\eta_2\,\mathbf{c}\alpha_2}{\ell_1\ell_2} + \frac{\mathbf{s}\alpha_1\,\eta_1\,\mathbf{s}\alpha_2\,\eta_2}{\ell_1\ell_2} + \frac{\eta_2^2}{2\ell_2^2} + \frac{\eta_1^2}{2\ell_1^2}. \end{aligned}$$

We emphasize that H depends on the functions $\alpha_1(t), \alpha_2(t), \dots$, whereas the expression on the right is a polynomial in the variables *vars*. There is a correspondence, but they are not equal, hence the $\longleftrightarrow$ symbol.

The geodesic equations can be written as follows:

$$
\begin{aligned}
\frac{d}{dt}\alpha_1 &= \frac{\cos(\alpha_1)\,p_2}{\ell_1} - \frac{p_1\sin(\alpha_1)}{\ell_1} \\
&\quad + \frac{\eta_2\cos(\alpha_1)\cos(\alpha_2)}{\ell_1\ell_2} + \frac{\eta_2\sin(\alpha_1)\sin(\alpha_2)}{\ell_1\ell_2} + \frac{\eta_1}{\ell_1^2}, \\
\frac{d}{dt}\alpha_2 &= \frac{\cos(\alpha_2)\,p_2}{\ell_2} - \frac{p_1\sin(\alpha_2)}{\ell_2} \\
&\quad + \frac{\eta_1\cos(\alpha_1)\cos(\alpha_2)}{\ell_1\ell_2} + \frac{\eta_1\sin(\alpha_1)\sin(\alpha_2)}{\ell_1\ell_2} + \frac{\eta_2}{\ell_2^2}, \\
\frac{d}{dt}\eta_1 &= \frac{\eta_1\sin(\alpha_1)\,p_2}{\ell_1} + \frac{p_1\cos(\alpha_1)\,\eta_1}{\ell_1} \\
&\quad + \frac{\eta_1\eta_2\sin(\alpha_1)\cos(\alpha_2)}{\ell_1\ell_2} - \frac{\eta_1\eta_2\cos(\alpha_1)\sin(\alpha_2)}{\ell_1\ell_2}, \\
\frac{d}{dt}\eta_2 &= \frac{\eta_2\sin(\alpha_2)\,p_2}{\ell_2} + \frac{p_1\cos(\alpha_2)\,\eta_2}{\ell_2} \\
&\quad - \frac{\eta_1\eta_2\sin(\alpha_1)\cos(\alpha_2)}{\ell_1\ell_2} + \frac{\eta_1\eta_2\cos(\alpha_1)\sin(\alpha_2)}{\ell_1\ell_2}, \\
\frac{d}{dt}p_1 &= 0, \quad \frac{d}{dt}p_2 = 0, \\
\frac{d}{dt}x_1 &= -\frac{\sin(\alpha_1)\,\eta_1}{\ell_1} - \frac{\sin(\alpha_2)\,\eta_2}{\ell_2} + p_1, \\
\frac{d}{dt}x_2 &= \frac{\eta_1\cos(\alpha_1)}{\ell_1} + \frac{\eta_2\cos(\alpha_2)}{\ell_2} + p_2.
\end{aligned}
\tag{11}
$$

We make note of a trivial, but important technical point that could be troublesome. The relevant computer tools (for example, *Basis, NormalForm* in Maple, *GroebnerBasis, PolynomialReduce* in Mathematica) are used to manipulate *polynomials* in the *variables*

$$vars = \{\alpha_1, \alpha_2, \eta_1, \eta_2, p_1, p_2, \mathbf{c}\alpha_1, \mathbf{c}\alpha_2, \mathbf{s}\alpha_1, \mathbf{s}\alpha_2\},$$

whereas the differentiations will be applied to *polynomials* in the *functions*

$$\{x_1(t), x_2(t), p_1(t), p_2(t), \cos(\alpha_1(t)), \cos(\alpha_2(t)), \sin(\alpha_1(t)), \sin(\alpha_2(t)), \eta_1(t), \eta_2(t)\}.$$

It is trivial to create tools *FtoP, PtoF* which translate one to the other.

We finally begin the computation. We will be interested in working with the condition $H - \frac{1}{2} = 0$, so that the projected curve $x = (x_1(t), x_2(t))$ is arc length parameterized. Thus, our ideal I will be generated by $Hpoly - \frac{1}{2}$, as well as the trivial trigonometric identities:

$$PB = \{Hpoly - \frac{1}{2}, \mathbf{c}\alpha_1{}^2 + \mathbf{s}\alpha_1{}^2 - 1, \mathbf{c}\alpha_2{}^2 + \mathbf{s}\alpha_2{}^2 - 1\}.$$

Using the appropriate tool, one computes its associated Gröbner basis *PGB*, allowing us to do polynomial calculations $\mod I$. We re-derive the formula for curvature: we have

$$\frac{d}{dt}x_1(t) \leftrightarrow -\frac{\mathbf{s}\alpha_I\,\eta_1}{\ell_1} - \frac{\mathbf{s}\alpha_2\,\eta_2}{\ell_2} + p_1, \quad \frac{d^2}{dt^2}x_1(t) \leftrightarrow -\frac{(\eta_1+\eta_2)\,(\mathbf{c}\alpha_I\,\eta_1 + \mathbf{c}\alpha_2\,\eta_2 + \ell_1 p_2)}{\ell_1^3},$$
$$\frac{d}{dt}x_2(t) \leftrightarrow \frac{\eta_1\,\mathbf{c}\alpha_I}{\ell_1} + \frac{\eta_2\,\mathbf{c}\alpha_2}{\ell_1} + p_2, \quad \frac{d^2}{dt^2}x_2(t) \leftrightarrow \frac{(\eta_1+\eta_2)\,(-\mathbf{s}\alpha_I\,\eta_1 - \mathbf{s}\alpha_2\,\eta_2 + \ell_1 p_1)}{\ell_1^3}.$$

As a result,

$$\left(\frac{d}{dt}x_1(t)\right)\left(\frac{d^2}{dt^2}x_2(t)\right) - \left(\frac{d^2}{dt^2}x_1(t)\right)\left(\frac{d}{dt}x_2(t)\right) \leftrightarrow \kappa poly_x = \frac{\eta_1}{\ell_1^2} + \frac{\eta_2}{\ell_1^2} \mod I.$$

We use κ_x to denote the curvature for the curve $x = (x_1(t), x_2(t))$, $\kappa poly_x$ the associated polynomial in terms of the variables *vars*. Note that this agrees with equation (5) .

As a brief aside, one can compute *mutatis mutandis* the curvatures of the curves y_1 and y_2 using this same technique, whereas computing by hand is somewhat messy. Here is the polynomial associated with the y_1 curvature, the y_2 formula is similar:

$$\kappa poly_{y_1} = \frac{2\,\mathbf{c}\alpha_I\,\mathbf{c}\alpha_2\,\eta_1}{\ell_1^2} + \frac{2\,\mathbf{c}\alpha_I\,p_2}{\ell_1} + \frac{2\eta_1\,\mathbf{s}\alpha_I\,\mathbf{s}\alpha_2}{\ell_1^2} - \frac{2p_I\,\mathbf{s}\alpha_I}{\ell_1} - \frac{\eta_1}{\ell_1^2} + \frac{\eta_2}{\ell_1^2}.$$

For computational purposes, this expression is the one we wish to work with. The reader is invited to manipulate the associated functions, and come up with a simpler expression.

We return to the elastica calculation, the case when $\ell_1 = \ell_2$. We wish to show that there exists a constant A such that $\kappa = \kappa_x$ satisfies

$$\frac{d^2}{dt^2}\kappa(t) = -\frac{\kappa(t)^3}{2} - A\kappa(t). \tag{12}$$

Define $\kappa_0, \kappa_1, \kappa_2$ as the polynomials associated with $\kappa, \dot{\kappa}, \ddot{\kappa}$ respectively. At each stage, differentiate a function, substitute the differential equations system (11), convert to a polynomial, and then compute and reduce via the Gröbner basis *GPB*. Substitute the derivatives in (12); one obtains an equation for A, with solution which has the corresponding polynomial:

$$Apoly = -\frac{\mathbf{c}\alpha_I\,p_2\eta_1}{\ell_1^3} - \frac{\mathbf{c}\alpha_2\,\eta_2 p_2}{\ell_1^3} + \frac{\eta_2 p_1\,\mathbf{s}\alpha_2}{\ell_1^3} - \frac{p_1^2}{\ell_1^2} - \frac{p_2^2}{\ell_1^2} + \frac{p_1\,\mathbf{s}\alpha_I\,\eta_1}{\ell_1^3} - \frac{\eta_2^2}{2\ell_1^4} - \frac{\eta_2\eta_1}{\ell_1^4} - \frac{\eta_1^2}{2\ell_1^4}.$$

To show that A is actually constant, differentiate and substitute the equations (11); the expression reduces to 0, as desired.

Essentially all results in this paper can be proved directly by hand in the equal length case, and proofs are given. Our results for unequal lengths depend heavily on the Gröbner basis reduction tools.

8 Bäcklund transformations of 1-soliton curves to 2-soliton curves

In this section, we recall an elementary contruction in curve theory. For our purposes, we only need the contruction for planar curves. We seek to answer the following question: given a curve γ, how can we construct a new curve $\tilde{\gamma}$, with a point-to-point correspondence between the new curves, such that

1. the segments representing the point-to-point correspondence have *constant* length;

2. the arc length parameterization is preserved via the correspondence?

The construction has various names: "bicycle correspondence", Bäcklund transformation (see [BLPT20, Tab17], and [RS02]).

Specifically, given $\gamma(t)$ with arc length parameter t, we construct a new curve $\tilde{\gamma}(t)$ via

$$\tilde{\gamma}(t) = \gamma(t) + L\,(\cos\beta(t)\,T(t) + \sin\beta(t)\,N(t)),$$

where L is constant and (T,N) is the Frenet frame along γ.

Note that condition 1) is automatically satisfied, with segment length L, but that t is initially only known to be the arc length parameter along γ, not $\tilde{\gamma}$. We ask when the t parameter in $\tilde{\gamma}$ is actually arc length along the second curve. Of course, this means that $\dot{\tilde{\gamma}}\cdot\dot{\tilde{\gamma}} - 1 = 0$, and implies the following restriction on β:

$$\left(\dot{\beta}(t) + \kappa(t)\right)\left(L\dot{\beta} + L\kappa(t) - 2\sin(\beta(t))\right) = 0,$$

where $\kappa(t)$ is the curvature of γ.

If the first factor vanishes, then $\dot{\beta} = -\kappa$, which corresponds to the displacement field $L(\cos\beta T + \sin\beta N)$ being constant along γ. Otherwise, we have the first order differential equation for β:

$$\frac{d}{dt}\beta(t) = -\kappa(t) + \frac{2\sin(\beta(t))}{L}$$

(this is the "bicycle" ODE, compare with Remark 5.2). Using the differential equation, one can easily compute that:

$$\begin{aligned}
\tilde{T} &= T(t)\cos(2\beta(t)) + N(t)\sin(2\beta(t))\,;\\
\tilde{N} &= -N(t)\cos(2\beta(t)) + T(t)\sin(2\beta(t))\,;\\
\tilde{\kappa} &= \kappa(t) - \frac{4\sin(\beta(t))}{L}.
\end{aligned}$$

If γ is an elastic curve, we can the describe the geometry of its Bäcklund transformation $\tilde{\gamma}$.

Theorem 6. *If γ is an elastic curve, then $\tilde{\gamma}$ is a 2-soliton.*

Proof. We are given the equations

$$\begin{aligned}
\left(\frac{d}{dt}\kappa(t)\right)^2 = -\kappa(t)^2 A - \frac{\kappa(t)^4}{4} - B,\;\; \frac{d^2}{dt^2}\kappa(t) = -\kappa(t)A - \frac{\kappa(t)^3}{2},\\
\tilde{\kappa} = \kappa(x) + \frac{4\sin(\beta(x))}{L},\;\; \frac{d}{dt}\beta(t) = -\kappa(t) - \frac{2\sin(\beta(t))}{L},
\end{aligned} \tag{13}$$

and we wish to conclude that $\tilde{\kappa}$ satisfies, for appropriate constants c_1, c_2, the equation

$$\kappa(t)\, c_1 + \left(-\frac{d^2}{dt^2}\kappa(t) - \frac{\kappa(t)^3}{2}\right) c_2 + \frac{5\kappa(t)^2 \left(\frac{d^2}{dt^2}\kappa(t)\right)}{2} + \frac{5\kappa(t)\left(\frac{d}{dt}\kappa(t)\right)^2}{2} + \frac{3\kappa(t)^5}{8} + \frac{d^4}{dt^4}\kappa(t) = 0, \tag{14}$$

of course, with κ replaced by $\tilde{\kappa}$.

We define $\kappa_0 = \tilde{\kappa}, \kappa_1 = \dot{\kappa}_0, \ldots \kappa_4 = \dot{\kappa}_3$. At each stage after the differentiation, we simplify the expression using (13); the resulting expressions are polynomial in $\cos\beta, \sin\beta$ with no β derivatives, and only involve polynomial expressions in κ and, at most, a term *linear* in $\dot{\kappa}$.

The coefficient of $\dot{\kappa}$ in the above expression is

$$\frac{4\left(\kappa L + 4\sin(\beta)\right)\left(\left(\frac{B}{8} + \frac{c_1}{4}\right)L^2 + A\right)}{L^3},$$

hence

$$c_1 = -\frac{B}{2} - \frac{4A}{L^2}.$$

Substituting this value back in our equation, we obtain

$$c_2 = -A + \frac{4}{L^2},$$

as needed. □

We saw in Section 6 that, in the equal length case, the path of the “front wheel" x is an elastica. We can now describe the geometry of the trajectories of the “rear points" y_1, y_2.

By construction, the paths of the rear points y_1, y_2 are in Bäcklund correspondence with the path of x (an elastica) with lengths equal to 2ℓ. Hence we can conclude:

Corollary 8.1. *The paths of y_1, y_2 are 2-soliton curves.*

Proof. By Theorem 6, the curvatures $\kappa_{y_1}, \kappa_{y_2}$, satisfy (14) with coefficients c_1, c_2:

$$c_1 = -\frac{B}{2} - \frac{4A}{L^2}, \quad c_2 = -A + \frac{4}{L^2},$$

where $L = 2\ell$, and A, B are related as in Lemma 6.1. □

9 Examples of tricycle geodesics

Example 9.1. Consider the special case when x traverses a unit circle: $\gamma(t) = t$. In this case, equations (10) can be solved exactly.

One type of solution is when $\alpha_2 = t - \pi/2$ and $\alpha_1(t)$ is any function satisfying $\dot{\alpha}_1 = \sin(t - \alpha_1)$. This solution corresponds, via lifting, to the special solution of the 1-linkage problem when the rear wheel of the bicycle is fixed at the center of the unit circle. Of course, by symmetry, there is a solution with $\alpha_1 = t - \pi/2$ and $\alpha_2(t)$ is any function satisfying $\dot{\alpha}_2 = \sin(t - \alpha_2)$.

A calculation that we omit shows that there are no other solutions to (10) with $\gamma(t) = t$.

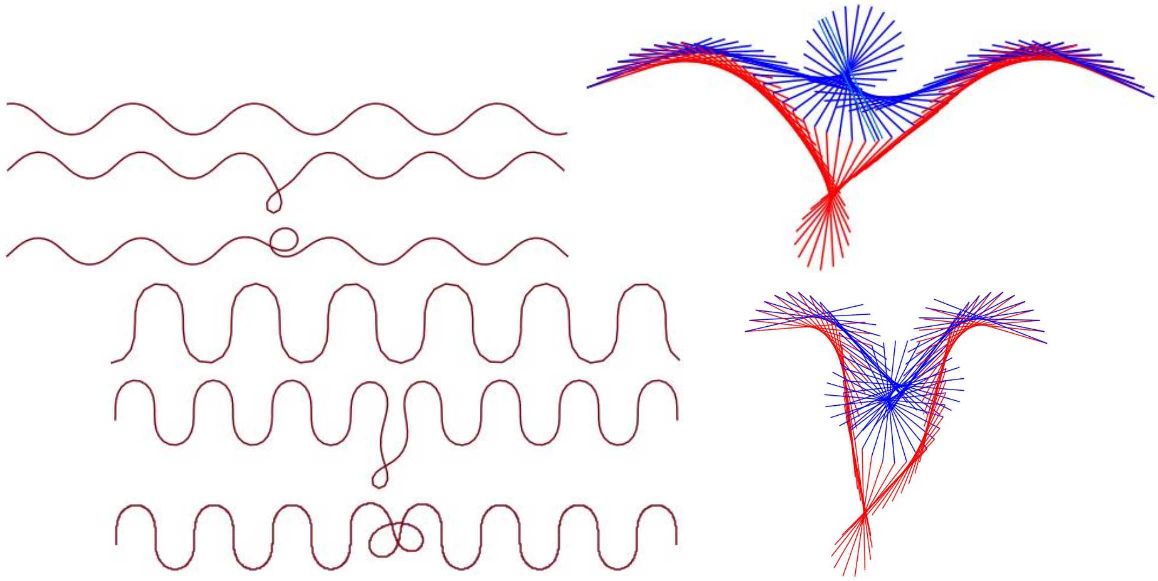

Figure 4: On the left, the trajectories of points x, y_1, and y_2 are shown, and on the right, the motion of the 2-linkage whose two links are colored red and blue. The values of k are 0.1 in the top graphs, and 0.707 in the bottom graphs, respectively. The trajectories on the right are moved apart for readability.

Example 9.2. Here, we generate initial conditions for equation (11) which produce various inflectional elasticae. We assume that $\ell_1 = \ell_2 = 1$. For convenience, the variables used here will (by slight abuse of notation) represent functions values at 0. For example, η_1 represents $\eta_1(0))$, $\mathbf{c}\alpha_1$ represents $\cos(\alpha_1(0)$, etc.

It is well known that the function $\nu = 2k\,\mathrm{cn}(t,k)$, $0 < k < 1$, is the formula for the curvature of inflectional elastica (the limiting cases of $k = 0$, 1 correspond to the line and Euler soliton respectively). Thus, we have $\nu(0) = 2k$ and $\dot{\nu}(0) = 0$. Also, we have

$$\ddot{\nu} + \frac{\nu^3}{2} + A_\nu \nu = 0,$$

where $A_\nu = -2k^2 + 1$.

On the other hand, in Section 7 we saw that the values of κ_x and its derivative at 0 are given by $\eta_1 + \eta_2$ and

$$\mathbf{c}\alpha_1\,\eta_1 p_1 + \mathbf{c}\alpha_2\,\eta_2 p_1 + \mathbf{s}\alpha_1\,\eta_1 p_2 + \mathbf{s}\alpha_2\,\eta_2 p_2,$$

respectively. In addition, we know that κ_x satisfies the differential equation

$$\ddot{\kappa} + \frac{\kappa^3}{2} + A_L \kappa = 0,$$

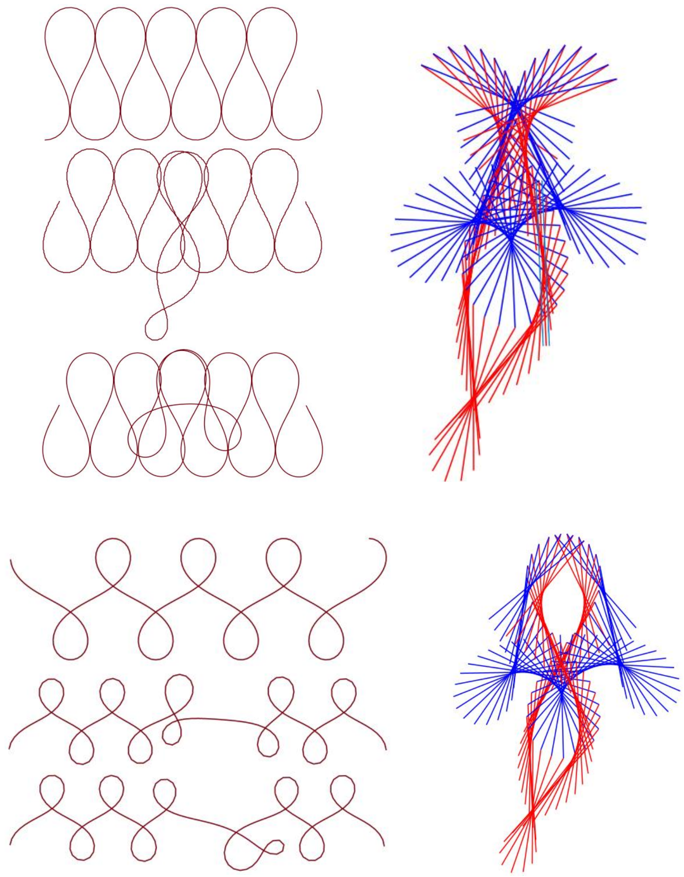

Figure 5: On the left, the trajectories of points x, y_1, and y_2 are shown, and on the right, the motion of the 2-linkage whose two links are colored red and blue. The values of k are 0.854 in the top graphs, and 0.95 in the bottom graphs, respectively. The trajectories on the right are moved apart for readability.

where A_L is given by (L stands for linkage)

$$\begin{aligned}A_L = &- \mathbf{c}\alpha_1\, p_2\eta_1 - \mathbf{c}\alpha_2\, \eta_2 p_2 + \eta_2 p_1\, \mathbf{s}\alpha_2 + p_1\, \mathbf{s}\alpha_1\, \eta_1 \\ &- \frac{1}{2}\eta_2^2 - \eta_2\eta_1 - p_1^2 - p_2^2 - \frac{1}{2}\eta_1^2.\end{aligned}$$

If we can find initial conditions satisfying

$$\begin{aligned}&\eta_1 + \eta_2 = 2k, \\ &\mathbf{c}\alpha_1\, \eta_1 p_1 + \mathbf{c}\alpha_2\, \eta_2 p_1 + \mathbf{s}\alpha_1\, \eta_1 p_2 + \mathbf{s}\alpha_2\, \eta_2 p_2 = 0, \\ &A_L = A_\nu\end{aligned}$$

with the additional condition that $H - \frac{1}{2} = 0$ (so that we have unit speed parameterization of the (x_1, x_2) curve), then the functions ν and κ_x satisfy the same differential equation with same initial conditions;

and, by the standard theorem of existence and uniqueness of solutions to ordinary differential equations (e.g.,[CL55]), the two functions are equal.

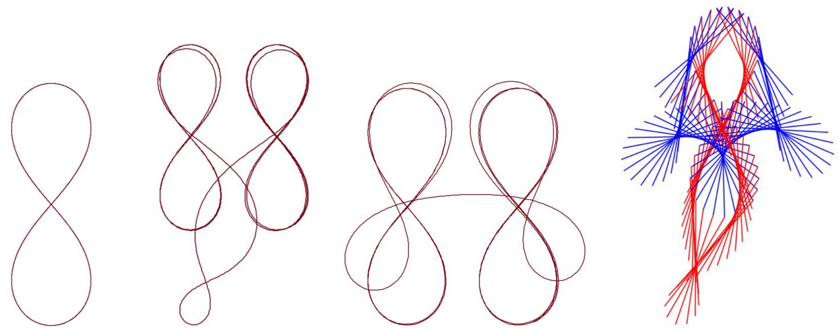

Figure 6: From left to right: the trajectories of points x, y_1, y_2 and the respective motion of the 2-linkage. Here $k = 0.909$.

There are many solutions to this set of equations; by judicious choice of α_1, p_1, p_2, we obtain the solution:

$$\alpha_1(0) = \arctan\left(\frac{4}{3}\right), \quad \alpha_2(0) = \arctan\left(\frac{4k^2 - 10k + 4}{3k^2 - 3}\right), \quad \eta_1(0) = \frac{5k^2 - 5}{5k - 4},$$
$$\eta_2(0) = \frac{5k^2 - 8k + 5}{5k - 4}, \quad p_1(0) = 1, p_2(0) = 0, \quad x_1(0) = 0, \quad x_2(0) = 0.$$

Inserting values $k = .1$, $.707$, $.854$, $.95$, and $.909$, one obtains the initial conditions which produce various different types of inflectional elasticae, see Figure 4, 5, and 6.

Note that, in the last case (Figure 6), the trajectory of point x is a closed curve, but the trajectories of points y_1 and y_2 are not. We do not know whether there exists a closed tricycling geodesic whose x-projection is an eight-shaped elastica (that is, the trajectories of all three points, x, y_1, y_2 are periodic).

10 The case of unequal length line segments and the re-appearance of 2-soliton curves

Before making any hypotheses about the geometry of the paths of x, y_1, y_2, we looked at some plots of these paths – see Figure 7.

It is immediately obvious that *none* of the three paths are elastica. However, given the results of the last section, it is plausible that these curves are 2-soliton curves. In this section, we prove that this is the case.

Theorem 7. *If $\ell_1 \neq \ell_2$, all three planar projections of a tricycling geodesic are 2-solitons.*

Proof. We begin with the x-curve. We know that its curvature is given by

$$\kappa_x = \frac{\eta_2(t)}{{\ell_2}^2} + \frac{\eta_1(t)}{{\ell_1}^2},$$

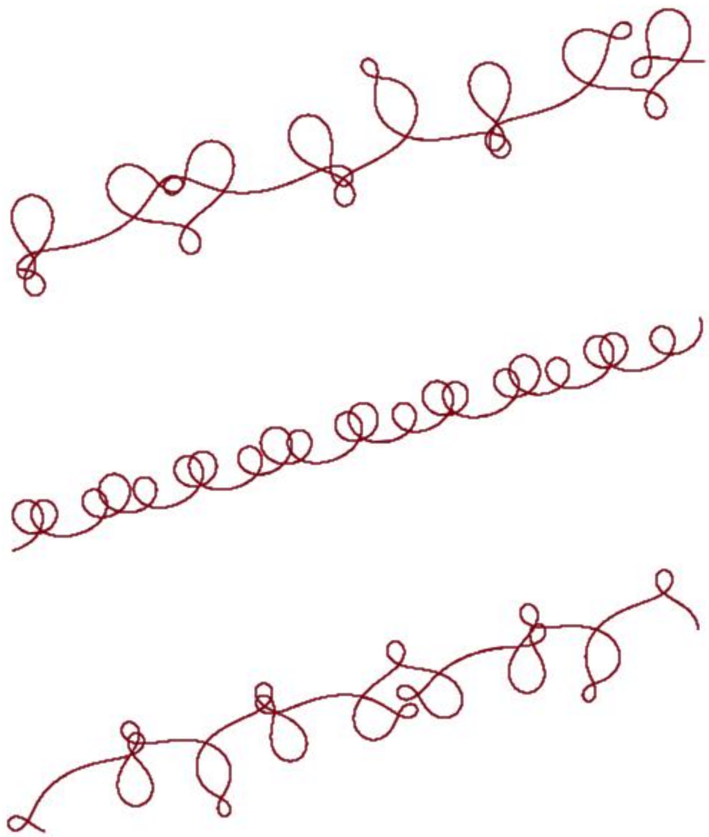

Figure 7: From top to bottom: the trajectories of points x, y_1, and y_2.

and we wish to show, as in the equal length case, that κ_x satisfies the differential equation

$$a\kappa(t) + b\left(-\frac{d^2}{dt^2}\kappa(t) - \frac{\kappa(t)^3}{2}\right) + \frac{5\kappa(t)^2\left(\frac{d^2}{dt^2}\kappa(t)\right)}{2} + \frac{5\kappa(t)\left(\frac{d}{dt}\kappa(t)\right)^2}{2} + \frac{3\kappa(t)^5}{8} + \frac{d^4}{dt^4}\kappa(t) = 0 \tag{15}$$

for appropriate a, b.

As before, we can set $\kappa_0 = \kappa_x, \kappa_1 = \dot{\kappa}_0, \ldots \kappa_4 = \dot{\kappa}_3$, again simplifying the expressions at each step using (11) and algebraic reduction, then replacing the derivatives of κ by the appropriate κ_i. However, this will simply give us one condition on the two unknowns a, b.

We remedy this situation by differentiating (15) to obtain a *fifth* order equation for κ_x, requiring an additional computation of $\kappa_5 = \dot{\kappa}_4$. After algebraic reduction, one obtains the polynomial representation for b:

$$bpoly = \frac{\mathbf{c}\alpha_1\, p_2 \eta_1}{\ell_1 \ell_2^2} + \frac{\mathbf{c}\alpha_2\, \eta_2 p_2}{\ell_1^2 \ell_2} - \frac{\eta_2 p_1\, \mathbf{s}\alpha_2}{\ell_1^2 \ell_2} + \frac{p_1^2}{2\ell_2^2} + \frac{p_2^2}{2\ell_2^2} - \frac{p_1\, \mathbf{s}\alpha_1\, \eta_1}{\ell_1 \ell_2^2} + \frac{p_1^2}{2\ell_1^2} + \frac{p_2^2}{2\ell_1^2} + \frac{\eta_2^2}{2\ell_1^2\ell_2^2} + \frac{\eta_1\eta_2}{\ell_1^2\ell_2^2} + \frac{1}{2\ell_2^2} + \frac{1}{2\ell_1^2} + \frac{\eta_1^2}{2\ell_1^2\ell_2^2}$$

and a considerably longer expression for a. One can check directly that a, b are constants of motion.

In anticipation of our discussion of the complete integrability of (11), we note that H, p_1, p_2 are constants of motion. Our configuration space is four-dimensional, so we could anticipate an addtional constant of motion, but here we discover two of them, b and a. We hypothesize that a is functionally dependent on b.

Thus, we augment our polynomial basis $PB_2 = PB \bigcup \{bpoly - \lambda\}$, corresponding to a larger ideal I_2, and use the appropriate tool to compute a new Gröbner basis GPB_2 for I_2. One obtains

$$apoly = \frac{p_1^2}{2\ell_1^2\ell_2^2} + \frac{p_2^2}{2\ell_1^2\ell_2^2} - \frac{\lambda^2}{2} + \frac{\lambda}{\ell_2^2} + \frac{\lambda}{\ell_1^2} - \frac{1}{2\ell_2^4} - \frac{1}{2\ell_1^2\ell_2^2} - \frac{1}{2\ell_1^4} \mod I_2.$$

Hence, by replacing λ with b in the previous formula, we see that a is in fact a quadratic function of b.

One can also check, using the same sort of computations, that $\kappa_{y_1}, \kappa_{y_2}$ both satisfy (15) with the *same* values of a and b just computed. □

In the discussion of complete integrability which follows, it will be convenient to use an alternative notation $\tilde{G} = b$.

11 Complete integrability of the 2-linkage geodesic equations

In this section we show that the geodesic equations of the 2-linkage (4) are completely integrable. We begin with the case of equal lengths.

Consider the cotangent bundle $T^*\mathcal{C}$ with its standard symplectic form

$$dx_1 \wedge dp_1 + dx_2 \wedge dp_2 + d\alpha_1 \wedge d\eta_1 + d\alpha_2 \wedge d\eta_2.$$

We have four integrals of motion

$$\begin{aligned} H &= \frac{1}{2}[(p_1 - \eta_1 \sin\alpha_1 - \eta_2 \sin\alpha_2)^2 + (p_2 + \eta_1\cos\alpha_1 + \eta_2\cos\alpha_2)^2], \ \ p_1, \ \ p_2, \\ G &= \frac{\ddot{\kappa}}{\kappa} + \frac{1}{2}\kappa^2 = p_1(p_1 - \eta_1\sin\alpha_1 - \eta_2\sin\alpha_2) + p_2(p_2 + \eta_1\cos\alpha_1 + \eta_2\cos\alpha_2) \\ &+ \frac{1}{2}(\eta_1 + \eta_2)^2, \end{aligned} \tag{16}$$

where we used (5) and (7).

Proposition 11.1. *Assume that $\ell_1 = \ell_2$. Then the above integrals Poisson commute and they are almost everywhere functionally independent.*

Proof. It is obvious that p_1 and p_2 are in involution with everything. That $\{H,G\}=0$ follows from the fact that G is constant along the trajectories of the Hamiltonian field with the Hamiltonian function H.

For the second statement, let

$$u_1=\eta_1\cos\alpha_1+\eta_2\cos\alpha_2,\ \ u_2=\eta_1\sin\alpha_1+\eta_2\sin\alpha_2.$$

The change of variables

$$(\alpha_1,\alpha_2,\eta_1,\eta_2)\mapsto(u_1,u_2,\eta_1,\eta_2)$$

has the Jacobian $\eta_1\eta_2\sin(\alpha_2-\alpha_1)$. Assume that this Jacobian doesn't vanish. Then

$$H=\frac{(p_1-v)^2+(p_2+u)^2}{2},\ \ G=p_1^2+p_2^2+p_2u_1-p_1u_2+\frac{(\eta_1+\eta_2)^2}{2}.$$

Since p_1 and p_2 are integrals, we need to see when ∇H and ∇G (as functions of u_1,u_2,η_1,η_2) are linearly dependent. The relevant 2×4 matrix is

$$\begin{pmatrix} p_2+u_1 & -p_1+u_2 & 0 & 0\\ p_2 & -p_1 & \eta_1+\eta_2 & \eta_1+\eta_2\end{pmatrix}.$$

If $\kappa=\eta_1+\eta_2\neq 0$, then this matrix does not have full rank if and only if $u_2=p_1,u_1=-p_2$, but this is impossible because then one would have $H=0$.

If $\kappa=0$, then the matrix does not have full rank if and only if $p_1u_1+p_2u_2=0$, that is,

$$p_1(\eta_1\cos\alpha_1+\eta_2\cos\alpha_2)+p_2(\eta_1\sin\alpha_1+\eta_2\sin\alpha_2)=0.$$

This is a hypersurface in the phase space.

To see the meaning of this condition, substitute η_1 and η_2 from (6) to arrive (as always, applying trigonometric identities) at $p_2\cos\gamma-p_1\sin\gamma=0$ or, in view of (8), $\dot\kappa=0$. □

We briefly discuss the case of unequal lengths. Here, we introduce a slightly different change of variables (which fails to work in the equal length case).

Define

$$\lambda_1=\frac{\cos(\alpha_1)\,\eta_1}{\ell_1}+\frac{\cos(\alpha_2)\,\eta_2}{\ell_2},\ \ \lambda_2=\frac{\sin(\alpha_1)\,\eta_1}{\ell_1}+\frac{\sin(\alpha_2)\,\eta_2}{\ell_2},$$
$$\mu_1=\frac{\eta_2\sin(\alpha_2)}{\ell_1}+\frac{\sin(\alpha_1)\,\eta_1}{\ell_2},\ \ \mu_2=\eta_2+\eta_1.$$

One can check that the determinant of the Jacobian of the transformation $(\alpha_1,\alpha_2,\eta_1,\eta_2)\to(\lambda_1,\lambda_2,\mu_1,\mu_2)$ is

$$\frac{\eta_1\eta_2\,(\cos(\alpha_1)\,\ell_1-\cos(\alpha_2)\,\ell_2)\left(\ell_1^2-\ell_2^2\right)}{\ell_1^3\ell_2^3}.$$

We can express our two Hamiltonians H and $\tilde G$ in terms of these new variables (WLOG, we henceforth make the assumption that $p_2=0$):

$$H=\frac{1}{2}\lambda_1^2-\lambda_1p_1+\frac{1}{2}\lambda_2^2+\frac{1}{2}p_1^2,\quad \tilde G=\frac{p_1^2}{2\ell_2^2}-\frac{\mu_1p_1}{\ell_2\ell_1}+\frac{p_1^2}{2\ell_1^2}+\frac{1}{2\ell_2^2}+\frac{1}{2\ell_1^2}+\frac{\mu_2^2}{2\ell_2^2\ell_1^2}.$$

To prove independence of the two functions H and $\tilde{G}$, we need to show that the matrix $[\nabla H, \nabla \tilde{G}]$ has full rank; but this matrix is easily computed to be:

$$\left[\begin{array}{cccc} \lambda_1 - p_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\frac{p_1}{\ell_2 \ell_1} & \frac{\mu_2}{{\ell_2}^2 {\ell_1}^2} \end{array}\right].$$

Ignoring the trivial case $p_1 = 0$ (where the x-path is a circle), we see that we have dependence only if both λ_2 and $\lambda_1 - p_1$ are simultaneously equal to zero.

12 Geometric interpretation of the constants of motion $G, \tilde{G}$. The planar filament equation

Recall that for the equal length case, we have a constant of motion G, whereas we had a constant of motion $\tilde{G}$ in the unequal case. One can check that if $\ell_1 = \ell_2 = \ell$, then $\tilde{G} - G = \frac{1}{\ell^2}$. In this section we discuss a geometric interpretation of $G, \tilde{G}$ in their respective cases.

First, assume that $\ell_1 = \ell_2 = 1$.

Fix a point of $Q \in T^*\mathcal{C}$, and consider the flow lines of the commuting Hamiltonian vector fields X_H and X_G through point Q, where H and G are as in (16).

Project these curves to the plane. The first curve is the trajectory $x(t)$, an elastica. Let $(T(t), N(t))$ be the Frenet frame along this curve. The tangent vectors to the second curve provide a vector field ξ along this elastica. This vector field can be written as $\xi = uT + vN$, where u and v are functions of t.

Theorem 8. *In the above notations, one has*

$$u(t) = 1 + g - \frac{1}{2}\kappa^2(t), \; v(t) = -\dot{\kappa}(t),$$

where $\kappa(t)$ is the curvature of the curve $x(t)$, and g is the value of the integral G at point Q.

Proof. The projection of the vector $X_{\tilde{G}}(Q)$ on the plane is the vector

$$\begin{aligned} \xi = (G_{p_1}, G_{p_2}) &= (2p_1 - \eta_1 \sin\alpha_1 - \eta_2 \sin\alpha_2, 2p_2 + \eta_1 \cos\alpha_1 + \eta_2 \cos\alpha_2) \\ &= (p_1 + \cos\gamma, p_2 + \sin\gamma), \end{aligned}$$

where the last equality is due to (3). One has

$$T = (\cos\gamma, \sin\gamma), N = (\sin\gamma, \cos\gamma),$$

and the linear equation $\xi = uT + vN$ has the solution

$$u = 1 + p_1 \cos\gamma + p_2 \sin\gamma, \;\; v = -p_1 \sin\gamma + p_2 \cos\gamma.$$

Use equations (7) and (8) and the definition of G to conclude:

$$u = 1 + \frac{\ddot{\kappa}}{\kappa} = 1 + g - \frac{1}{2}\kappa^2(t), \;\; v(t) = -\dot{\kappa}(t),$$

as claimed. ☐

Thus the vector field ξ is a linear combination of the unit tangent field T and the planar filament vector field $\frac{1}{2}\kappa^2 T + \dot{\kappa}N$, see [LP92]. Note that the flow of this field takes the elastic curves to elastic curves, in agreement with the fact that the flows of the Hamiltonians H and G commute.

If $\ell_1 \neq \ell_2$, we have the analogous statement, proved by the computer algebra techniques already discussed.

Theorem 9. *One has*

$$u(t) = \tilde{g} - \frac{1}{2}\kappa^2(t), \;\; v(t) = -\dot{\kappa}(t),$$

where $\kappa(t)$ is the curvature of the curve $x(t)$, and $\tilde{g}$ is the value of the integral $\tilde{G}$ at point Q.

13 Concluding remarks

Inspired by papers [ABLD$^+$21, BJT23], we have obtained a 2-linkage model which exhibits the fundamental observation in those references: the vertices of the moving linkage sweep out curves of interesting geometry. Amusingly, our problem, involving a completely non-integrable distribution, is completely integrable.

It is clear that there is much more worth studying related to this problem. Here are some open questions:

- Elastic curves have been studied in the context of spherical and hyperbolic geometry, see [LS84]; additionally, analogues of the planar filament flow exist in those geometries as well, see [LP94]. This suggests that our 2-linkage problem could be amenable to study is these contexts too.

- In this paper, we have by no means solved the geodesic equations, we have only shown a certain striking property of the projection of the solutions to the plane. Can these equations be solved exactly?

- Paper [BJT23] extends the work in [ABLD$^+$21] to 1-linkages in R^n. We expect the results of our paper to be extendable to higher dimensional cases as well.

- Isospectrality is a nearly universal property of integrable systems, although this can take some work to uncover. To give one example, the problem of integrability of geodesics on the ellipsoid goes back to Jacobi, but the isospectral framework for the problem was discovered by Moser [Mos80]. Does such a framework appear in the case of our problem?

- The starting point of our work was a different sub-Riemmanian problem, concerning the geodesic motion of a 2-linkage xyz subject to the following non-holonomic constraint: the velocity of point y is aligned with the segment xy, and the velocity of the point z with the segment zy, see Figure 8. The lengths of the two segments are fixed, and the length a horizontal curve is defined as the length of its x-projection to the plane.

  Rather than modeling bicycle, this is a model of tractor trailer, which is a well studied topic; see, e.g., [TMS95] and the references therein. Unfortunately, we did not obtain meaningful results in this case, but we think that it merits a study. See Figure 9 for an example of the planar projections of a geodesic.

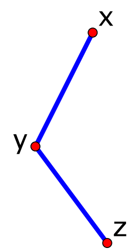

Figure 8: Another 2-linkage.

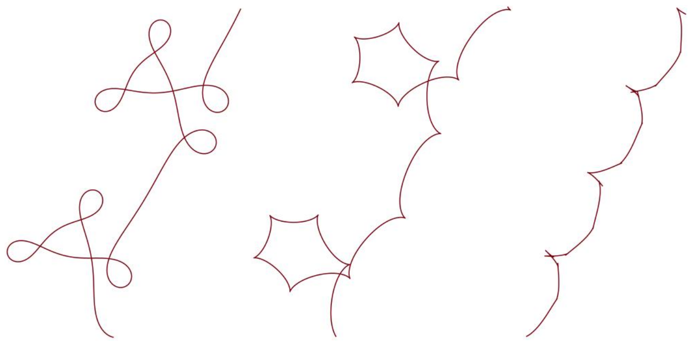

Figure 9: The x, y, and z projections of a geodesic.

- Why stop at 2-linkages? There are obvious n-linkage problems to consider, where the non-holonomic constraints apply to the midpoints of each segment, and one wants to extremize the length of the trajectory of either vertex (they have the same length).

  In the case of three or more segments, the *topology* of the linkage is not unique. In the case of 3-linkages, one can have a "linear" linkage, or a "spokewheel" linkage, see Figure 10.

Figure 10: Two topologically different 3-linkages.

Our last figures give examples of the trajectories in the "linear" and "spokewheel" cases. Perhaps one or both of these problems is integrable, as well extensions to linkages with four or more links. See Figure 11, 12.

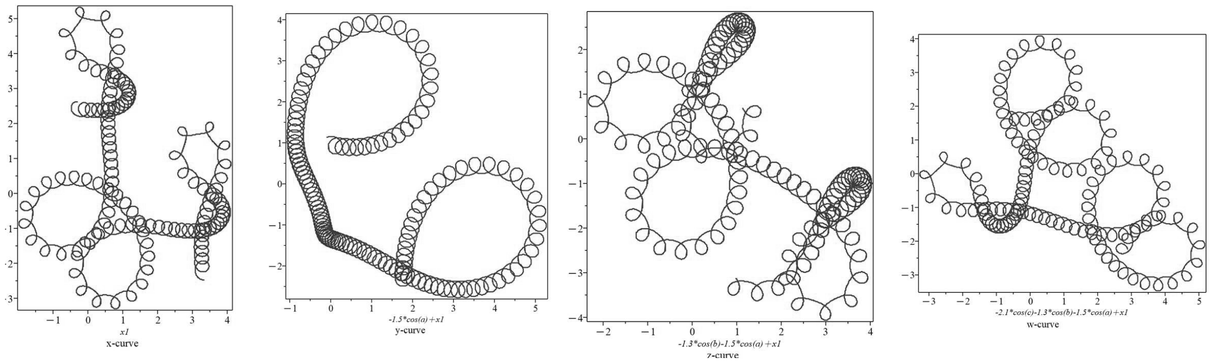

Figure 11: The trajectories of four vertices of a “linear" 3-linkage, the case of three unequal lengths.

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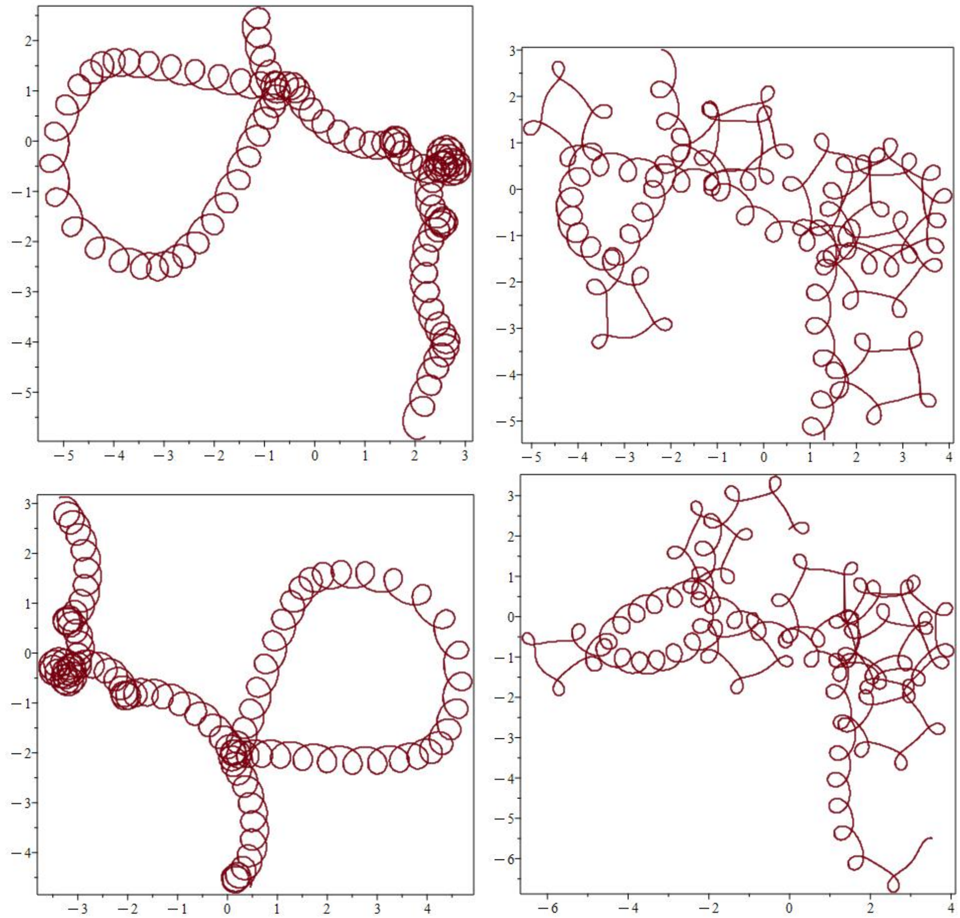

Figure 12: The trajectories of four vertices of a “spokewheel" 3-linkage, the case of three unequal lengths. The first trajectory is of that of the middle point.

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