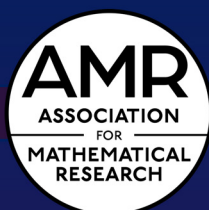
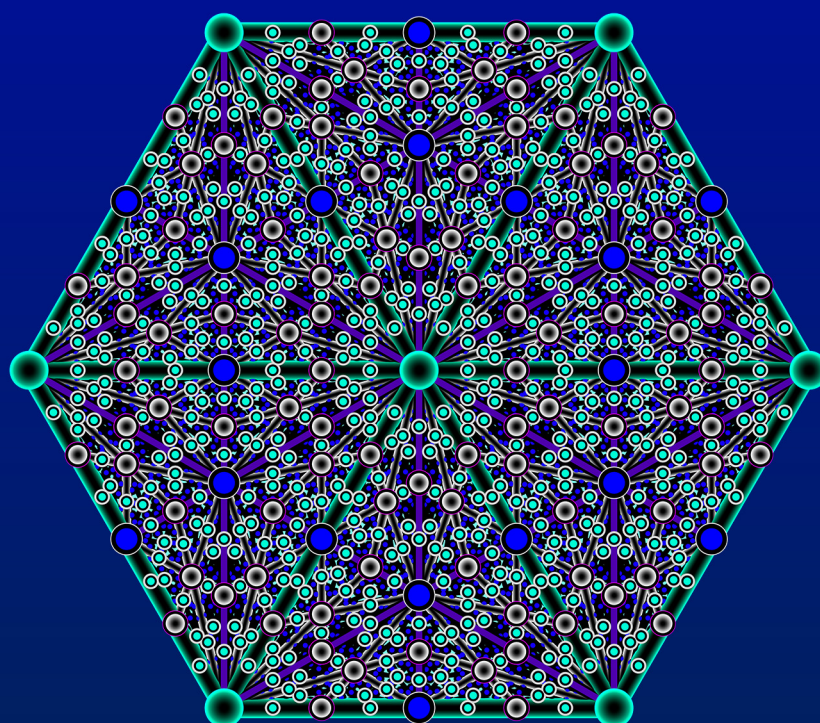


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Ebert’s Asymmetric Three Person Three Color Hat Game

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Abstract:

We generalize Ebert’s Hat Problem for three persons and three colors. All players guess simultaneously the color of their own hat observing only the hat colors of the other players. It is also allowed for each player to pass: no color is guessed. The team wins if at least one player guesses his or her hat color correct and none of the players has an incorrect guess. This paper studies Ebert’s hat problem, where the probabilities of the colors may be different (asymmetric case). Our goal is to maximize the probability of winning the game and to describe winning strategies. In this paper we use the notion of an adequate set. The construction of adequate sets is independent of underlying probabilities and we can use this fact in the analysis of the asymmetric case. Another point of interest is the fact that computational complexity using adequate sets is much less than using standard methods.

Key words and phrases: asymmetric hat game, coding theory, computational complexity

1 Introduction

Hat puzzles were formulated at least since Martin Gardner’s 1961 article [Gar61]. They have got an impulse by Todd Ebert in his Ph.D. thesis in 1998 [Ebe98]. Buhler [Buh02] stated: “It is remarkable that a purely recreational problem comes so close to the research frontier”. Also articles in The New York Times [Rob01], Die Zeit [Blu01] and abcNews [Pou01] about this subject got broad attention. This paper studies a generalization of Ebert’s hat problem. The original problem was symmetric: each color has probability $\frac{1}{3}$. We consider the asymmetric case: three distinguishable players are randomly fitted with a colored hat (three colors available), where the probabilities of getting a specific color may be different, but known to all the players. All players guess simultaneously the color of their own hat observing only

the hat colors of the other two players. It is also allowed for each player to pass: no color is guessed. The team wins if at least one player guesses his or her hat color correctly and none of the players has an incorrect guess. Our goal is to maximize the probability of winning the game and to describe winning strategies. The symmetric two color hat problem (probability $\frac{1}{2}$ for each color) with $N = 2^k - 1$ players is solved in [EMV03], using Hamming codes, and with $N = 2^k$ players in [CHLL97] using extended Hamming codes. Burke et al. [BGK02] try to solve the symmetric hat problem with $N = 3, 4, 5, 7$ players using genetic programming. Their conclusion: The N -prisoners puzzle (alternative names: Hat Problem, Hat Game) gives evolutionary computation and genetic programming a new challenge to overcome. Lenstra and Seroussi [LS05] show that in the symmetric case of two hat colors, and for any value of N , playing strategies are equivalent to binary covering codes of radius one. Combining the result of Lenstra and Seroussi with Tables for Bounds on Covering Codes [Ké], we get:

N	2	3	4	5	6	7	8	9
$K(N, 1)$	2	2	4	7	12	16	32	62

$K(N, 1)$ is smallest size of a binary covering code of radius 1. Maximum probability for Ebert's symmetric two color Hat Game is $1 - \frac{K(N,1)}{2^N}$. Lower bound on $K(9, 1)$ was found in 2001 by Östergård-Blass, the upper bound in 2005 by Östergård. Krzywkowski [Krz10] describes applications of the hat problem and its variations, and their connections to different areas of science. Krzywkowski [Krz11] and Guo et al. [GKRT06] gives an optimal solution of the symmetric three person three color hat problem. Tantipongpipat [Tan14] proves an optimal strategy for Ebert's hat game with three players and more than two hat colors. Johnson [Joh] ends his presentation with an open problem: If the hat colors are not equally likely, how will the optimal strategy be affected? We will answer this question and our method gives also interesting results in the symmetric case. In section 2 we define an adequate set. In section 3 we obtain results for the asymmetric three person three color Hat Game, where each player has the same probabilities to get a specific colored hat. In section 4 we get new results for the symmetric three person three color Hat Game by using the adequate set method. In all situations all players know the underlying probabilities of each player.

2 Adequate sets

In this section we have N players and q colors. The N persons in our game are distinguishable, so we can label them from 1 to N . We label the q colors $0, 1, \dots, q-1$. The probabilities of the colors are fixed and known to all players. The probability that color i will be on a hat is p_i ($i \in \{0, 1, \dots, q-1\}$, $\sum_{i=0}^{q-1} p_i = 1$). Each possible configuration of the hats can be represented by an element of

$$B = \{b_1 b_2 \dots b_N \mid b_i \in \{0, 1, \dots, q-1\}, i = 1, 2, \dots, N\}.$$

The S-code represents what the N different players sees. Player i sees q -ary code $b_1 \dots b_{i-1} b_{i+1} \dots b_N$ with decimal value

$$s_i = \sum_{k=1}^{i-1} b_k \cdot q^{N-k-1} + \sum_{k=i+1}^N b_k \cdot q^{N-k},$$

a value between 0 and $q^{N-1} - 1$.

Let S be the set of all S-codes:

$$S = \{s_1 s_2 \dots s_N | s_i = \sum_{k=1}^{i-1} b_k \cdot q^{N-k-1} + \sum_{k=i+1}^N b_k \cdot q^{N-k}, b_i \in \{0, 1, \dots, q-1\}, i = 1, 2, \dots, N\}.$$

Each player has to make a choice out of $q+1$ possibilities: 0='guess color 0', 1='guess color 1', ..., $q-1$ = 'guess color $q-1$ ', q ='pass'.

We define a decision matrix $D = (a_{i,j})$ where $i \in \{1, 2, \dots, N\}$ (players); $j \in \{0, 1, \dots, q^{N-1} - 1\}$ (S-code of a player); $a_{i,j} \in \{0, 1, \dots, q\}$. The meaning of $a_{i,j}$ is: player i sees S-code j and takes decision $a_{i,j}$ (guess a color or pass). We observe the total probability (sum) of our guesses.

For each $b_1 b_2 \dots b_N$ in B with n_i times color i ($i = 0, 1, \dots, q-1, \sum_{i=0}^{q-1} n_i = N$) and S-code player i :

$$s_i = \sum_{k=1}^{i-1} b_k \cdot q^{N-k-1} + \sum_{k=i+1}^N b_k \cdot q^{N-k}$$

we have:

IF $a_{1,s_1} \in \{q, b_1\}$ AND $a_{2,s_2} \in \{q, b_2\}$ AND ... AND $a_{N,s_N} \in \{q, b_N\}$ AND

NOT ($a_{1,s_1} = a_{2,s_2} = \dots = a_{N,s_N} = q$) THEN sum = sum + $p_0^{n_0} \cdot p_1^{n_1} \dots p_{q-1}^{n_{q-1}}$.

Any choice of the $a_{i,j}$ in the decision matrix determines which CASES $b_1 b_2 \dots b_N$ have a positive contribution to sum (we call it a GOOD CASE) and which CASES don't contribute positive to sum (we call it a BAD CASE).

Definition 1. Let $A \subset B$. A is adequate to $B - A$ if for each q -ary element x in $B - A$ there are $q-1$ elements in A which are equal to x up to one fixed q -ary position.

Theorem 1. BAD CASES are adequate to GOOD CASES.

Proof. Any GOOD CASE has at least one $a_{i,j}$ not equal to q . Let this specific $a_{i,j}$ have value b_{i_0} . Then our GOOD CASE generates $q-1$ BAD CASES by only changing the value b_{i_0} in any value of $0, 1, \dots, q-1$ except b_{i_0} . $\square$

The definition of adequate set is the same idea as the concept of strong covering, introduced by Lenstra and Seroussi [LS05]. The number of elements in an adequate set will be written as das (dimension of adequate set). Adequate sets are generated by an adequate set generator (ASG). For an implementation of an ASG in VBA/Excel see Appendix A.

Theorem 2. Each adequate set generates a decision matrix D .

Proof. For each element in the adequate set:

- Determine the q -ary representation $b_1 b_2 \dots b_N$.
- Calculate S-codes $s_i = \sum_{k=1}^{i-1} b_k \cdot q^{N-k-1} + \sum_{k=i+1}^N b_k \cdot q^{N-k}$ ($i=1, \dots, N$).
- For each player i : fill decision matrix with $a_{i,s_i} = b_i$ ($i=1, \dots, N$), where a cell may contain several values.

Matrix D is filled with BAD COLORS. We can extract the GOOD COLORS by considering all $a_{i,j}$ with $q - 1$ BAD COLORS and then choose the only missing color. In all situations with less than $q - 1$ BAD COLORS we pass. When there is an $a_{i,j}$ with q BAD COLORS all colors are bad, so the first option is to pass. But when we choose any color, we get a situation with $q - 1$ BAD COLORS. So in case of q BAD COLORS we are free to choose any color or pass. The code for pass is q , but in our decision matrices we prefer a blank, which supports readability. The code for 'any color or pass will do' is defined $q + 1$, but in our decision matrices we prefer a $\star$. $\square$

We call the procedure just described DMG (Decision Matrix Generator). An implementation of a DMG in VBA/Excel can be found in Appendix B.

3 Asymmetric Hat Game with three players and three colors

In this section we obtain results for asymmetric Hat Game with three players and three colors with hat probabilities p, q, r ($p + q + r = 1, pqr > 0$). Without loss of generality we suppose $p \geq q \geq r$.

3.1 Optimal winning probabilities

We first execute the program ASG with values $p=0.7, q=0.2, r=0.1$, $\text{das}=2,3,\dots,11$, which yields no adequate sets. Setting $\text{das}=12$ gives a collection of 324 adequate sets. There are 3 optimal adequate sets, each with probability 0.242:

4	5	7	8	9	13	14	16	17	18	20	24	0.242
1	2	8	12	13	15	16	20	21	22	24	25	0.242
3	6	8	10	11	13	14	19	20	22	23	24	0.242

So maximal winning probability of this game is $1-0.242=0.758$. The probability of an adequate set is a function $\varphi = \sum p^a q^b r^{3-a-b}$, where the summation is over all elements of the adequate set and a and b are the number of zero's and one's in the ternary representation of each element of the adequate set. For each of the three adequate sets just found, we have:

$$\varphi = qr^2 + 2q^2r + q^3 + 3pr^2 + 2pqr + pq^2 + p^2r + p^2q$$

(e.g. the first element of the first adequate set, 4, is ternary 011 and contributes pq^2 to φ ; this process is implemented in matrix $a(i, j)$ in ASG, see appendix A). We describe this as follows:

00	01	02	03	10	11	12	20	21	30
0	1	2	1	3	2	1	1	1	0

In the first line ab means: we have $p^a q^b r^{3-a-b}$ and in the second line we find the coefficients of $p^a q^b r^{3-a-b}$. We call (0121321110) the pattern of the adequate set.

Definition 2. A pattern $P1$ is dominant over pattern $P2$ when the φ value of $P1$ is equal or less than the φ value of $P2$.

The number of adequate sets when $\text{das} \geq 12$ is shown in the following table (use ASG with different values of das).

12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
324	5832	50814	237816	670464	1194345	1389018	1097388	613251	250272	76086	17334	2925	351	27	1

We have to automate the search process of dominant patterns.

Lemma 3. The b -pattern $b_1 b_2 \dots b_{10}$, where $b_1 = b(0,0)$ $b_2 = b(0,1)$ $b_3 = b(0,2)$ $b_4 = b(0,3)$ $b_5 = b(1,0)$ $b_6 = b(1,1)$ $b_7 = b(1,2)$ $b_8 = b(2,0)$ $b_9 = b(2,1)$ $b_{10} = b(3,0)$ with $\dim(b) = \sum b_i$ is dominant over the a -pattern $a_1 a_2 \dots a_{10}$, where $a_1 = a(0,0)$ $a_2 = a(0,1)$ $a_3 = a(0,2)$ $a_4 = a(0,3)$ $a_5 = a(1,0)$ $a_6 = a(1,1)$ $a_7 = a(1,2)$ $a_8 = a(2,0)$ $a_9 = a(2,1)$ $a_{10} = a(3,0)$ with $\dim(a) = \sum a_i$, $\dim(a) = \dim(b)$,

$c_j = b_j - a_j$ ($j = 1, 2, \dots, 10$), $m_j = \max(0, -c_j)$ when:

$$c_1 \geq 0$$

$$c_1 + c_2 \geq m_3 + m_5$$

$$(c_1 + c_2 + c_3 \geq m_4 + m_5 + m_6) \vee (c_1 + c_2 + c_3 \geq m_4 \wedge c_5 \geq 0 \wedge c_5 + c_6 \geq 0)$$

$$(c_1 + c_2 + c_3 + c_4 \geq m_5 + m_6) \vee (c_1 + c_2 + c_3 + c_4 \geq 0 \wedge c_5 \geq 0 \wedge c_5 + c_6 \geq 0)$$

$$c_1 + c_2 + c_5 \geq m_3$$

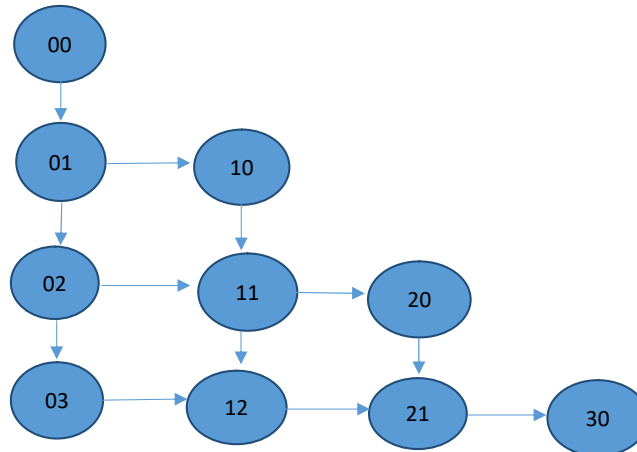
$$(c_1 + c_2 + c_3 + c_5 + c_6 \geq m_4 + m_7 + m_8) \vee (c_1 + c_2 + c_3 + c_5 + c_6 \geq m_8 \wedge c_4 \geq 0 \wedge c_4 + c_7 \geq 0)$$

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 \geq m_8$$

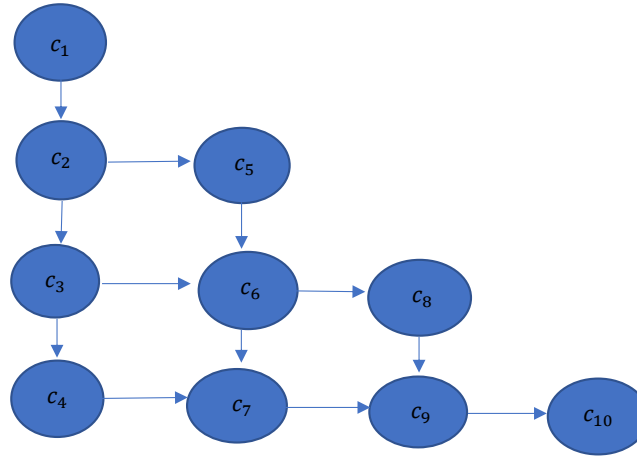
$$(c_1 + c_2 + c_3 + c_5 + c_6 + c_8 \geq m_4 + m_7) \vee (c_1 + c_2 + c_3 + c_5 + c_6 + c_8 \geq 0 \wedge c_4 \geq 0 \wedge c_4 + c_7 \geq 0)$$

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 \geq 0$$

Proof. The dominance relations between the atoms of φ are (dominance is represented by an arrow):



The b -pattern is dominant over the a -pattern when in each vertex of the next directed graph there is enough compensation:



To illustrate our proof we concentrate on cell 6 (with c_6). Maximum flow to cell 6 (inclusive cell 6) is $c_1 + c_2 + c_3 + c_5 + c_6$. But eventually cell 3 has to compensate a possible negative value c_4 in cell 4. Also cell 6 must compensate potential negative values c_7 and c_8 in cells 7 and 8. This leads to:

$$c_1 + c_2 + c_3 + c_5 + c_6 \geq m_4 + m_7 + m_8$$

To make a stronger procedure we remark that when $c_4 \geq 0 \wedge c_4 + c_7 \geq 0$ then compensation for cell 8 is sufficient. We get:

$$(c_1 + c_2 + c_3 + c_5 + c_6 \geq m_4 + m_7 + m_8) \vee$$

$$(c_4 \geq 0 \wedge c_4 + c_7 \geq 0 \wedge c_1 + c_2 + c_3 + c_5 + c_6 \geq m_8)$$

The same reasoning for all other cells, where cell 10 may be omitted (we have $\text{das} \geq \text{das}$). $\square$

We implemented this dominance test in the procedure `dom()` in ASG. We can also use this procedure when the two sets have different dimensions. When $e = \dim(a) - \dim(b) > 0$ we add e elements in cell 1 of pattern b and then we apply the foregoing procedure.

Lemma 4. $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ dominates all adequate sets, where

$$\varphi_1 = (0121321110).$$

$$\varphi_2 = (1210141110).$$

$$\varphi_3 = (1331000301).$$

$$\varphi_4 = (1331121110).$$

Proof. We execute ASG with $\text{das} = 12$ and different values of p, q, r . When $p = 0.7, q = 0.2, r = 0.1$ we find three optimal adequate sets with pattern $\varphi_1 = (0121321110)$. Using $p = \frac{1}{2}, q = \frac{1}{3}, r = \frac{1}{6}$, we get optimal adequate sets with pattern $\varphi_2 = (1210141110)$. For $p = 0.35, q = 0.33, r = 0.32$ we obtain the optimal pattern $\varphi_3 = (1331000301)$. Executing ASG with `dom( $\varphi_1$ ), dom( $\varphi_2$ ), dom( $\varphi_3$ )` activated (see appendix A), we get no adequate sets: $\{\varphi_1, \varphi_2, \varphi_3\}$ dominates all adequate sets when $\text{das} = 12$. We get the same result when $\text{das} = 13$. Setting $\text{das} = 14$, any values of p, q, r and `dom( $\varphi_1$ ), dom( $\varphi_2$ ), dom( $\varphi_3$ )` activated with $e = 2$, we get 10 adequate sets (out of 50814), all dominated by or equal to

$\varphi_4 = (1331121110)$. In Appendix C we show that $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ dominates all adequate sets when $\text{das} > 14$.

Conclusion: $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ dominates all adequate sets. $\square$

Our task is now to minimize (adequate set, BAD CASES) the value of φ , given values of p, q, r . Let $\Psi_i = 1 - \varphi_i$ ($i = 1, 2, 3, 4$). Ψ_i is the probability of winning the game.

Lemma 5. $\Psi_1 = p(1 - 2r^2) + (1 - p)^2(p + r)$

$$\Psi_2 = 1 + p^2 r + 2pr^2 + p^2 - p - r$$

$$\Psi_3 = 3p(1 - p - pr)$$

$$\Psi_4 = p(p^2 - 2p + 2)$$

Proof.

$$\Psi_1 = 1 - \varphi_1 = 1 - (qr^2 + 2q^2r + q^3 + 3pr^2 + 2pqr + pq^2 + p^2r + p^2q).$$

Eliminate q to get the desired result for Ψ_1 . Analogue for Ψ_2, Ψ_3 and Ψ_4 . $\square$

We notice that because of $p + q + r = 1$ and $p \geq q \geq r > 0$, we have:

$p \geq \frac{1}{3}$ and $1 - 2p \leq r \leq \frac{1-p}{2}$ (see also Figure 1).

Lemma 6. $\{\varphi_2, \varphi_4\}$ dominates φ_1

Proof. $\Psi_2 - \Psi_1 = (1 - p - 2r) \left[(1 - p)^2 - 2pr \right] \geq 0$, when $0 < r \leq \min\left\{\frac{(1-p)^2}{2p}, \frac{1-p}{2}\right\}$.

$\Psi_4 - \Psi_1 = r[2pr - (1 - p)^2] \geq 0$ when $\frac{(1-p)^2}{2p} \leq r \leq \frac{1-p}{2}$.

So: φ_1 is dominated by $\{\varphi_2, \varphi_4\}$ (see also Figure 1). $\square$

Lemma 7. Ψ_4 is an optimal winning probability when

$$\frac{(1-p)^2}{2p} \leq r \leq \frac{1-p}{2} \quad \left(\frac{1}{2} \leq p < 1\right)$$

Proof. $\Psi_4 - \Psi_2 = (1 - p - r)[2pr - (1 - p)^2] = q[2pr - (1 - p)^2]$

$\Psi_4 - \Psi_3 = p(p^2 + p - 1 + 3pr) > \frac{3}{2}p[2pr - (1 - p)^2]$ and $\frac{(1-p)^2}{2p} \leq r \leq \frac{1-p}{2}$ implies $\frac{1}{2} \leq p < 1$ (see also Figure 1). $\square$

Let α be the solution of $\frac{1-2p}{2p} = \frac{1-p}{2}$ with $\frac{1}{3} \leq p < 1$, so $\alpha = \frac{3-\sqrt{5}}{2}$.

Lemma 8. Ψ_3 is an optimal winning probability when

$$\left[1 - 2p \leq r \leq \frac{1-p}{2} \quad \left(\frac{1}{3} \leq p \leq \alpha\right)\right] \vee \left[1 - 2p \leq r \leq \frac{1-2p}{2p} \quad \left(\alpha \leq p \leq \frac{1}{2}\right)\right]$$

Proof. We have (see also Figure 1):

If $\frac{1}{3} \leq p \leq \alpha$ then $r \leq \frac{1-p}{2} < \frac{1-p-p^2}{3p}$.

If $\alpha \leq p \leq \frac{1}{2}$ then $r \leq \frac{1-2p}{2p} < \frac{1-p-p^2}{3p}$.

So when $\frac{1}{3} \leq p \leq \frac{1}{2}$ we have: $\Psi_3 - \Psi_4 = -p(p^2 + p - 1 + 3pr) > 0$.

$\Psi_3 - \Psi_2 = (1 - 2p - r)(2pr + 2p - 1) \geq 0$ when $r \leq \frac{1-2p}{2p}$.

The result follows from $\frac{1-p}{2} \leq \frac{1-2p}{2p}$ when $\frac{1}{3} \leq p \leq \alpha$. $\square$

Lemma 9. Ψ_2 is an optimal winning probability when

$$\left[\frac{1-2p}{2p} \leq r \leq \frac{1-p}{2} \quad (\alpha \leq p \leq \frac{1}{2}) \right] \vee \left[0 < r \leq \frac{(1-p)^2}{2p} \quad (\frac{1}{2} \leq p < 1) \right]$$

Proof. $\Psi_2 - \Psi_4 = (r+p-1)[2pr - (1-p)^2] = -q[2pr - (1-p)^2]$.

$\Psi_2 - \Psi_3 = (2p+r-1)(2pr+2p-1)$.

If $p \geq \frac{1}{2}$ then $2p+r-1 \leq 0$ and $2pr+2p-1 \leq 0$ so $\Psi_2 \geq \Psi_3$. When $p \geq \frac{1}{2}$ and $r \leq \frac{(1-p)^2}{2p}$ we have $\Psi_2 \geq \Psi_4$.

When $\alpha \leq p \leq \frac{1}{2}$ we have $\frac{1-2p}{2p} \leq r \leq \frac{1-p}{2} \leq \frac{(1-p)^2}{2p}$, so: $\Psi_2 \geq \Psi_4$ and $\Psi_2 \geq \Psi_3$.

See also Figure 1. □

Theorem 10. The next table and graphics gives a description of optimal probabilities for asymmetric three person three color Hat Game:

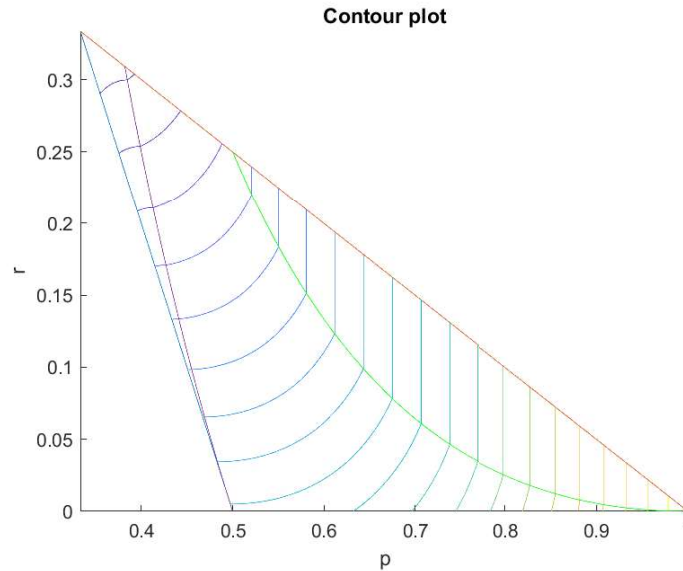
Set	Region	Optimal probability
A	$\frac{(1-p)^2}{2p} \leq r \leq \frac{1-p}{2} \quad (\frac{1}{2} \leq p < 1)$	Ψ_4
B	$\frac{1-2p}{2p} \leq r \leq \frac{1-p}{2} \quad (\alpha \leq p \leq \frac{1}{2}) \vee 0 < r \leq \frac{(1-p)^2}{2p} \quad (\frac{1}{2} \leq p < 1)$	Ψ_2
C	$1-2p \leq r \leq \frac{1-p}{2} \quad (\frac{1}{3} \leq p \leq \alpha) \vee 1-2p \leq r \leq \frac{1-2p}{2p} \quad (\alpha \leq p \leq \frac{1}{2})$	Ψ_3

$$\alpha = \frac{3-\sqrt{5}}{2}.$$

$$\Psi_2 = 1 + p^2r + 2pr^2 + p^2 - p - r.$$

$$\Psi_3 = 3p(1-p-pr).$$

$$\Psi_4 = p(p^2 - 2p + 2).$$



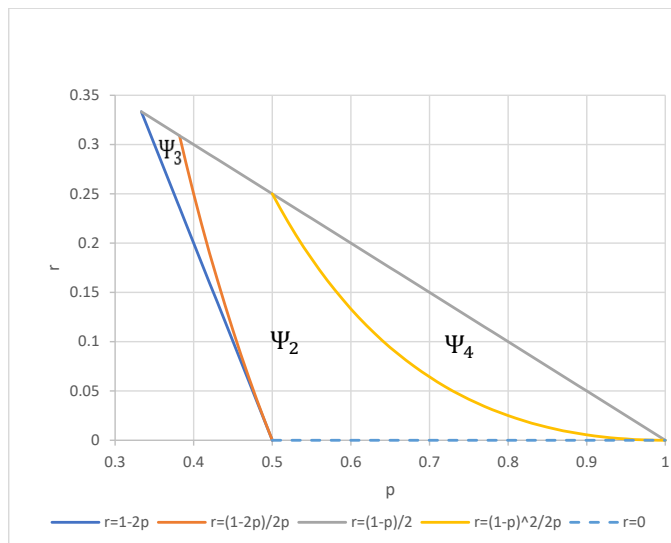
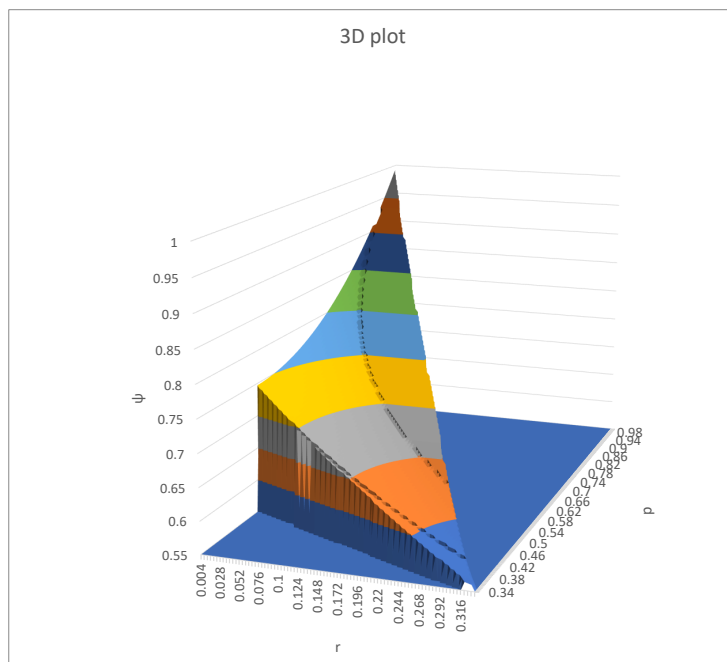


Figure 1: Optimal probabilities



Proof. Use Lemma's 6-9. □

We find all solutions of the asymmetric three person *two* color Hat Game by taking $r = 0$:

Theorem 11. *If $p > \frac{1}{2}$ then we have optimal probability $1 + p^2 - p$ with optimal decision matrix:*

00	01	10	11
0			1
	0	1	
	0	1	

If $p = q = \frac{1}{2}$ then we have optimal probability $\frac{3}{4}$ and two optimal decision matrices: the one already found when $p > \frac{1}{2}$ and the well-known:

00	01	10	11
1			0
1			0
1			0

Proof. If $p > \frac{1}{2}$ then we are in B with optimal probability $\Psi_2 = 1 + p^2 - p$ and we get the optimal decision matrix using the result of region B in section 3.2 and deleting color 2.

If $p = q = \frac{1}{2}$ then we have to do with sets B and C . We get $\Psi_2 = \Psi_3 = \frac{3}{4}$ and a second optimal decision matrix by deleting color 2 in the decision matrix of region C in section 3.2. $\square$

The same result is obtained in [vU24].

We investigate the behavior of the optimal probability Ψ_{opt} when $p \rightarrow 1$:

Theorem 12. $\Psi_{opt} = 1 - (1 - p) + (1 - p)^2 + \mathcal{O}([1 - p]^3) \quad (p \rightarrow 1)$

Proof. When $\frac{(1-p)^2}{2p} \leq r \leq \frac{1-p}{2}$ we have:

$$\Psi_{opt} = \Psi_4 = p(p^2 - 2p + 2) = 1 - (1 - p) + (1 - p)^2 - (1 - p)^3.$$

When $0 \leq r \leq \frac{(1-p)^2}{2p}$ then $\Psi_{opt} = 1 - p + p^2 + p^2r + 2pr^2 - r$, where

$$p^2r + 2pr^2 - r \leq 2pr^2 + (1 - p^2)r \leq \frac{(1-p)^4}{2p} + \frac{(1-p^2)(1-p)^2}{2p} = \frac{(1-p)^3}{p} = \frac{(1-p)^3}{1+(p-1)} = (1-p)^3 + \mathcal{O}([1-p]^4) \quad (p \rightarrow 1).$$

$\square$

Remark.

If $p \rightarrow 1$ then a simple strategy is: player 1 always guess the color with probability p and the other players always pass. But this simple strategy is not optimal:

$$\Psi_{opt} = p + (1 - p)^2 + \mathcal{O}([1 - p]^3) \quad (p \rightarrow 1).$$

Also surprising is that the behavior of q and r is relevant: regions A and B give two different optimal strategies.

3.2 Optimal strategies

Definition 3. Two adequate sets are isomorphic when there is a permutation of the three players which projects one set to another.

Theorem 13. Optimal strategies for three person, three color, hat game are:

Region A:

00	01	02	10	11	12	20	21	22
0				*	*		*	*
	0	0		0	0		0	0
	0	0		0	0		0	0

Region B:

00	01	02	10	11	12	20	21	22
0				1	1		1	1
	0	0	1				0	0
	0	0	1				0	0

Region C:

00	01	02	10	11	12	20	21	22
1				0	0		0	0
1				0	0		0	0
1				0	0		0	0

Proof. We start with region A. We take a point in A (e.g. $p = 0.7$, $q = 0.2$, $r = 0.1$). Ψ_4 is winner in A, so das=14. Execute the program ASG with das=14 and we find three optimal adequate sets:

4	5	7	8	9	13	14	16	17	18	22	23	25	26
1	2	12	13	14	15	16	17	21	22	23	24	25	26
3	6	10	11	13	14	16	17	19	20	22	23	25	26

These sets are isomorphic: cycles (13) and (23) will project the first set to the second and third one (using ternary code). Program DMG with das=14 gives the optimal decision matrix belonging to the first adequate set in region A. Region B: We consider a point in B (e.g. $p = \frac{1}{2}$, $q = \frac{1}{3}$, $r = \frac{1}{6}$), execute the program ASG with das=12 (Ψ_2 is winner) and we get 3 optimal adequate sets:

4	5	7	8	9	11	15	18	22	23	25	26
1	2	7	12	14	15	17	19	21	23	24	26
3	5	6	10	11	16	17	19	20	21	25	26

The three sets are isomorphic (again (13) and (12) will do). DMG gives the optimal decision matrix belonging to the first adequate set of region B. Region C: das=12 (Ψ_3 is winner). Using e.g. $p = 0.35$, $q = 0.33$, $r = 0.32$ we get one optimal adequate set when executing the program ASG with das=12:

0	2	6	13	14	16	17	18	22	23	25	26
---	---	---	----	----	----	----	----	----	----	----	----

Procedure DMG gives the optimal decision matrix for region C. □

3.3 Computational complexity

We consider the number of strategies to be examined to solve the hat problem with N players and q colors. Each of the N players has q^{N-1} possible situations to observe and in each situation there are $q+1$ possible guesses. So we have $((q+1)^{q^{N-1}})^N$ possible strategies. Krzywkowski [14] shows that it suffices to examine $((q+1)^{q^{N-1}-1})^N$ strategies. The adequate set method has to deal with $\{i_1, i_2, \dots, i_{das}\}$ with $0 \leq i_1 < i_2 < \dots < i_{das} \leq q^N - 1$.

The number of strategies for fixed das is the number of subsets of dimension das of $\{0, 1, \dots, q^N - 1\}$:

$\binom{q^N}{das}$. But we have to test all possible values of das . So the correct expression is:

$\sum_{das} \binom{q^N}{das} = 2^{(q^N)}$. The power of the adequate set method in the asymmetric 3 person, 3 color game is shown in the next table of computational complexity:

brute force	Krzywkowski	adequate set method
1,80144E+16	2,81475E+14	134217728

4 Symmetric Hat Game

In this section we focus on the symmetric Hat Game with three players and three colors.

4.1 Symmetric three color three person Hat Game

Definition 4. Two adequate sets are equivalent when they induce the same probability function.

Definition 5. The order of an adequate set is the number of equivalent adequate sets.

Theorem 14. When $p = q = r = \frac{1}{3}$ the optimal probability is $\frac{5}{9}$ and we have a collection of 77 non-isomorphic optimal decision matrices, each of order 1, 3 or 6. We give one matrix of each order: Order 1, adequate set $\{0\}$

00	01	02	10	11	12	20	21	22
2	2		2	2				1
2	2		2	2				1
2	2		2	2				1

Order 3, adequate set $\{4\}$

00	01	02	10	11	12	20	21	22
0				2	2		2	2
	0	0		0	0	1		
	0	0		0	0	1		

Order 6, adequate set $\{4\}$

00	01	02	10	11	12	20	21	22
0				2	2		2	2
	0	0		0	0	1		
	0	0		0	0	2		

Proof. Running ASG with any values of p, q, r gives 324 adequate sets. When $p = q = r = \frac{1}{3}$ all these sets are optimal with probability $\frac{5}{9}$. We run ASG with $p = 0.71, q = 0.23, r = 0.06$ (we don't want contamination of the probabilities). We sort the adequate sets by probability and find 75 equivalent sets of order 1, 3, 6 or 12 (see Appendix D).

order	count	order * count
1	6	6
3	36	108
6	31	186
12	2	24
total	75	324

Each set of order 12 can be split in two sets of order 6 which are not isomorphic: count number of non passes for each player, which gives a 663 or 555 pattern. Each set of order 3 is isomorphic to 2 equivalent sets of order 3 and each set of order 6 is isomorphic to 5 equivalent sets of order 6, which can be verified by permutations of the players. In total we have 77 non-isomorphic adequate sets. All 324 decision matrices are generated by DMG. $\square$

5 Conclusion

The use of adequate sets makes our object of investigation tractable by reducing the computational complexity with a factor greater than 10^5 . Three special cases are interesting.

The special case $r = 0$. Our game degenerates to a two color problem with known results.

The special case when p approaches 1. We get two optimal decision matrices which are superior to an intuitive optimal solution.

The symmetric case: We obtain all optimal solutions.

6 Related Topics

The relation with coding theory is well known (see e.g. [CHLL97] and [LS05]).

A broader view give relations between:

- coding theory and discrete tomography (see e.g. [DW06], where new approaches are based on modeling and coding theory) .
- collaborative game theory and cryptography (see e.g. [KEAGW18], where game theory is a suitable framework for the design of crypto-cloud computing system).
- modeling and inverse problems (see e.g. [KWAG4], cooperative interval games).

A Adequate Set Generator

```

Public k1, k2, k3, k4, k5, k6, k7, k8, k9, k10, e As Integer
Public dominant As Integer
Private Declare Function SetThreadExecutionState _
    Lib "Kernel32.dll" _
    (ByVal esFlags As Long) _
    As Long 'used in Sub KeepPowerOn() for long during programs
Sub dominant_adequate_set_generator()
Dim a() As Integer: Dim c() As Integer: Dim check() As Integer: Dim d() As Integer
Dim hit() As Integer: Dim i() As Integer: Dim j() As Integer: Dim neq() As Integer
Dim is_in_AS As Boolean
p = 0.7: q = 0.2: r = 0.1 'probability of three colors
n = 3 'three players
h = 3 ^ n - 1 '3^n elements in B
das = 12 'dimension of adequate set
ReDim a(0 To 3, 0 To 3): ReDim j(1 To das, 1 To n) As Integer
ReDim hit(0 To h, 1 To das) As Integer: ReDim d(1 To n) As Integer
ReDim i(1 To das) As Integer: ReDim c(0 To h, 1 To n) As Integer
ReDim check(0 To h) As Integer: ReDim neq(1 To n) As Integer
KeepPowerOn
" for each number from 0 to h: calculate ternary digits
For k = 0 To h
    g = k
    For Z = 1 To n: c(k, Z) = g Mod 3: g = g \ 3: Next Z
Next k
x = 0 'x is row number in Excel
'brute force for all potential adequate sets:
For i1 = 0 To h + 1 - das: i(1) = i1: For i2 = i1 + 1 To h + 2 - das: i(2) = i2
For i3 = i2 + 1 To h + 3 - das: i(3) = i3: For i4 = i3 + 1 To h + 4 - das: i(4) = i4
For i5 = i4 + 1 To h + 5 - das: i(5) = i5: For i6 = i5 + 1 To h + 6 - das: i(6) = i6
For i7 = i6 + 1 To h + 7 - das: i(7) = i7: For i8 = i7 + 1 To h + 8 - das: i(8) = i8
For i9 = i8 + 1 To h + 9 - das: i(9) = i9: For i10 = i9 + 1 To h + 10 - das: i(10) = i10

```

```

For i11 = i10 + 1 To h + 11 - das: i(11) = i11: For i12 = i11 + 1 To h + 12 - das: i(12) = i12

' fill adequate set matrix j:
For k = 1 To das
    g = i(k)
    For Z = 1 To n: j(k, Z) = g Mod 3: g = g \ 3: Next Z
Next k

' check on adequate set property:
For k = 0 To h
    check(k) = 0
    is_in_AS = False 'is not an element of AS=Adequate Set
    neq(1) = 0: neq(2) = 0: neq(3) = 0
    For m = 1 To das
        hit(k, m) = 0
        For t = 1 To 3
            If c(k, t) = j(m, t) Then hit(k, m) = hit(k, m) + 1
        Next t
        If hit(k, m) = 3 Then is_in_AS = True 'is element of Adequate Set
        If hit(k, m) = 2 And c(k, 1) <> j(m, 1) Then neq(1) = neq(1) + 1 'not equal for player 1
        If hit(k, m) = 2 And c(k, 2) <> j(m, 2) Then neq(2) = neq(2) + 1
        If hit(k, m) = 2 And c(k, 3) <> j(m, 3) Then neq(3) = neq(3) + 1
    Next m
    If neq(1) = 2 Or neq(2) = 2 Or neq(3) = 2 Or is_in_AS Then check(k) = 1 ' minimal 2 hits at same
place
Next k

State = 1 'potential adequate set
For k = 0 To h
    State = State * check(k)
Next k

Sum = 0
If State = 1 Then 'State=1 means adequate set
    For t = 0 To 3

```

```

For u = 0 To 3
    a(t, u) = 0 ' a(t,u) counts the number of  $p^t q^u r^{(3-t-u)}$  in adequate set
Next u
Next t
For k = 1 To das
    g = i(k)
    For Z = 1 To n: d(Z) = g Mod 3: g = g \ 3: Next Z
    count1 = 0: count2 = 0
    For t = 1 To n
        If d(t) = 0 Then count1 = count1 + 1
        If d(t) = 1 Then count2 = count2 + 1
    Next t
    Sum = Sum + p ^ count1 * q ^ count2 * r ^ (n - count1 - count2)
    a(count1, count2) = a(count1, count2) + 1
Next k
End If
k1 = a(0, 0): k2 = a(0, 1): k3 = a(0, 2): k4 = a(0, 3): k5 = a(1, 0)
k6 = a(1, 1): k7 = a(1, 2): k8 = a(2, 0): k9 = a(2, 1): k10 = a(3, 0)
dominant = 1
'k1..k10 is potential dominant (code 1); if dominated then switch: dominant=0;
If das >= 12 Then
    e = das - 12
    'Call dom(e, 1, 2, 1, 3, 2, 1, 1, 1, 0)
    'Call dom(e + 1, 2, 1, 0, 1, 4, 1, 1, 1, 0)
    'Call dom(e + 1, 3, 3, 1, 0, 0, 0, 3, 0, 1)
End If

If das >= 14 Then
    e = das - 14
    Call dom(e + 1, 3, 3, 1, 1, 2, 1, 1, 1, 0)
End If

```

```

If das >= 15 Then
  e = das - 15
  Call dom(e, 1, 2, 1, 3, 5, 2, 0, 0, 1)
  Call dom(e + 1, 1, 1, 1, 2, 6, 2, 0, 0, 1)
  Call dom(e, 2, 2, 0, 3, 4, 3, 0, 0, 1)
  Call dom(e + 1, 2, 1, 0, 2, 5, 3, 0, 0, 1)
End If

If das >= 16 Then
  e = das - 16
  Call dom(e, 2, 3, 1, 3, 4, 2, 0, 0, 1)
  Call dom(e + 1, 2, 2, 1, 2, 5, 2, 0, 0, 1)
  Call dom(e, 3, 3, 0, 3, 3, 3, 0, 0, 1)
  Call dom(e + 1, 3, 2, 0, 2, 4, 3, 0, 0, 1)
End If

If das >= 17 Then
  e = das - 17
  Call dom(e + 1, 3, 3, 1, 2, 4, 2, 0, 0, 1)
End If

If State = 1 And dominant = 1 Then ' dominant adequate set
  x = x + 1 'next row in Excel sheet: show elements of adequate set
  For k = 1 To das
    Cells(x, k) = i(k)
  Next k
  Cells(x, 3 + das) = Sum
End If

Next i12: Next i11: Next i10: Next i9: Next i8: Next i7: Next i6: Next i5: Next i4: Next i3: Next i2: Next
i1

```

EBERT'S ASYMMETRIC THREE PERSON THREE COLOR HAT GAME

```

End Sub

Sub dom(h1 As Integer, h2 As Integer, h3 As Integer, h4 As Integer, h5 As Integer, h6 As Integer, h7 As Integer, h8 As Integer, h9 As Integer, h10 As Integer)

Dim z1, z2, z3, z4, z5, z6, z7, z8, z9, m3, m4, m5, m6, m7, m8 As Integer

z1 = h1 - k1: z2 = h2 - k2: z3 = h3 - k3: z4 = h4 - k4: z5 = h5 - k5: z6 = h6 - k6: z7 = h7 - k7: z8 = h8 - k8: z9 = h9 - k9

If z3 < 0 Then m3 = -z3 Else m3 = 0: If z4 < 0 Then m4 = -z4 Else m4 = 0

If z5 < 0 Then m5 = -z5 Else m5 = 0: If z6 < 0 Then m6 = -z6 Else m6 = 0

If z7 < 0 Then m7 = -z7 Else m7 = 0: If z8 < 0 Then m8 = -z8 Else m8 = 0

c = 0

If z1 >= 0 Then c = c + 1 '1=00

If z1 + z2 >= m3 + m5 Then c = c + 1 '2=01

If z1 + z2 + z3 >= m4 + m5 + m6 Or (z1 + z2 + z3 >= m4 And z5 >= 0 And z5 + z6 >= 0) Then c = c + 1 '3=02

If z1 + z2 + z3 + z4 >= m5 + m6 Or (z1 + z2 + z3 + z4 >= 0 And z5 >= 0 And z5 + z6 >= 0) Then c = c + 1 '4=03

If z1 + z2 + z5 >= m3 Then c = c + 1 '5=10

If z1 + z2 + z3 + z5 + z6 >= m4 + m7 + m8 Or (z1 + z2 + z3 + z5 + z6 >= m8 And m4 >= 0 And m4 + m7 >= 0) Then c = c + 1 '6=11

If z1 + z2 + z3 + z4 + z5 + z6 + z7 >= m8 Then c = c + 1 '7=12

If z1 + z2 + z3 + z5 + z6 + z8 >= m4 + m7 Or (z1 + z2 + z3 + z5 + z6 + z8 >= 0 And m4 >= 0 And m4 + m7 >= 0) Then c = c + 1 '8=20

If z1 + z2 + z3 + z4 + z5 + z6 + z7 + z8 + z9 >= 0 Then c = c + 1 '9=21

If c = 9 Then dominant = 0

End Sub

Sub KeepPowerOn()

Const ES_SYSTEM_REQUIRED As Long = &H1

Const ES_DISPLAY_REQUIRED As Long = &H2

Const ES_CONTINUOUS As Long = &H80000000

Dim PrevState As Long

PrevState = SetThreadExecutionState(ES_CONTINUOUS Or ES_DISPLAY_REQUIRED Or ES_SYSTEM_REQUIRED)

Application.OnTime Now() + TimeValue("00:01:00"), "KeepPowerOn", , True

End Sub

```

B Decision Matrix generator

```

Sub decision_matrix_generator_n3_q3()
Dim a0() As Integer: Dim a1() As Integer: Dim a2() As Integer:
Dim c() As Integer: Dim check() As Integer: Dim d() As Integer
Dim hit() As Integer: Dim i() As Integer: Dim j() As Integer
Dim neq() As Integer: Dim is_in_AS As Boolean
n = 3: h = 3 ^ n - 1: v = 3 ^ (n - 1) - 1: das = 12
ReDim a0(1 To n, 0 To v) As Integer: ReDim a1(1 To n, 0 To v) As Integer
ReDim a2(1 To n, 0 To v) As Integer: ReDim c(0 To h, 1 To n) As Integer
ReDim check(0 To h) As Integer: ReDim d(1 To n) As Integer
ReDim hit(0 To h, 1 To das) As Integer: ReDim i(1 To das) As Integer
ReDim j(1 To das, 1 To n) As Integer: ReDim neq(1 To n) As Integer
' for each number from 0 to h put ternary digits in matrix c:
For k = 0 To h
    g = k
    For Z = 1 To n: c(k, Z) = g Mod 3: g = g \ 3: Next Z
Next k
x = -4 ' x: row in Excel where result is displayed
For i1 = 0 To h + 1 - das: i(1) = i1: For i2 = i1 + 1 To h + 2 - das: i(2) = i2
For i3 = i2 + 1 To h + 3 - das: i(3) = i3: For i4 = i3 + 1 To h + 4 - das: i(4) = i4
For i5 = i4 + 1 To h + 5 - das: i(5) = i5: For i6 = i5 + 1 To h + 6 - das: i(6) = i6
For i7 = i6 + 1 To h + 7 - das: i(7) = i7: For i8 = i7 + 1 To h + 8 - das: i(8) = i8
For i9 = i8 + 1 To h + 9 - das: i(9) = i9: For i10 = i9 + 1 To h + 10 - das: i(10) = i10
For i11 = i10 + 1 To h + 11 - das: i(11) = i11: For i12 = i11 + 1 To h + 12 - das: i(12) = i12
' ternary digits for adequate set:
For k = 1 To das
    g = i(k)
    For Z = 1 To n: j(k, Z) = g Mod 3: g = g \ 3: Next Z
Next k
' check on adequate set property:
For k = 0 To h
    check(k) = 0: is_in_AS = False

```

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```

neq(1) = 0: neq(2) = 0: neq(3) = 0

For m = 1 To das
    hit(k, m) = 0

    For t = 1 To 3
        If c(k, t) = j(m, t) Then hit(k, m) = hit(k, m) + 1
    Next t

    If hit(k, m) = 3 Then is_in_AS = True
    If hit(k, m) = 2 And c(k, 1) <> j(m, 1) Then neq(1) = neq(1) + 1
    If hit(k, m) = 2 And c(k, 2) <> j(m, 2) Then neq(2) = neq(2) + 1
    If hit(k, m) = 2 And c(k, 3) <> j(m, 3) Then neq(3) = neq(3) + 1
Next m

If neq(1) = 2 Or neq(2) = 2 Or neq(3) = 2 Or is_in_AS Then check(k) = 1 ' three colors, minimal 2 hits
at same place
Next k
State = 1
For k = 0 To h
    State = State * check(k)
Next k
If State = 1 Then 'State=1: adequate set
    x = x + n + 2
    For k = 1 To das
        Cells(x, k + 2) = i(k)
    Next k
    For S = 1 To n
        For k = 0 To v
            a0(S, k) = 0 'counts number of 0's in cell (s,k)
            a1(S, k) = 0
            a2(S, k) = 0
        Next k
    Next S
    For k = 1 To das

```

```
g = i(k)
```

```
For Z = n To 1 Step -1: d(Z) = g Mod 3: g = (g - d(Z)) / 3: Next Z
```

```
'd(1)d(2)d(3) is ternary representation of g
```

```
'update a0, a1 and a2:
```

```
If d(1) = 0 Then a0(1, 3 * d(2) + d(3)) = a0(1, 3 * d(2) + d(3)) + 1
```

```
If d(1) = 1 Then a1(1, 3 * d(2) + d(3)) = a1(1, 3 * d(2) + d(3)) + 1
```

```
If d(1) = 2 Then a2(1, 3 * d(2) + d(3)) = a2(1, 3 * d(2) + d(3)) + 1
```

```
If d(2) = 0 Then a0(2, 3 * d(1) + d(3)) = a0(2, 3 * d(1) + d(3)) + 1
```

```
If d(2) = 1 Then a1(2, 3 * d(1) + d(3)) = a1(2, 3 * d(1) + d(3)) + 1
```

```
If d(2) = 2 Then a2(2, 3 * d(1) + d(3)) = a2(2, 3 * d(1) + d(3)) + 1
```

```
If d(3) = 0 Then a0(3, 3 * d(1) + d(2)) = a0(3, 3 * d(1) + d(2)) + 1
```

```
If d(3) = 1 Then a1(3, 3 * d(1) + d(2)) = a1(3, 3 * d(1) + d(2)) + 1
```

```
If d(3) = 2 Then a2(3, 3 * d(1) + d(2)) = a2(3, 3 * d(1) + d(2)) + 1
```

```
Next k
```

```
For S = 1 To n
```

```
For k = 0 To v
```

```
If a0(S, k) + a1(S, k) + a2(S, k) <= 1 Then Cells(x + S, k + 1) = "" 'pass : blank
```

```
If a0(S, k) * a1(S, k) > 0 And a2(S, k) = 0 Then Cells(x + S, k + 1) = 2 'not 0 and not 1
```

```
If a0(S, k) * a2(S, k) > 0 And a1(S, k) = 0 Then Cells(x + S, k + 1) = 1
```

```
If a1(S, k) * a2(S, k) > 0 And a0(S, k) = 0 Then Cells(x + S, k + 1) = 0
```

```
If a0(S, k) * a1(S, k) * a2(S, k) > 0 Then Cells(x + S, k + 1) = "          *" 'any guess or pass will do
```

```
Next k
```

```
Next S
```

```
End If 'end of State=1
```

```
Next i12: Next i11: Next i10: Next i9: Next i8: Next i7: Next i6: Next i5: Next i4: Next i3: Next i2: Next i1
```

```
End Sub
```

C Dominance when $\text{das} > 14$

Running ASG with $\text{das}=15$ and procedure $\text{dom}()$ activated for $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ with $e = 3$, we get 70 sets. Sorting by probability gives 12 different probability classes, all dominated by or equal to 4 sets:

0	1	2	1	3	5	2	0	0	1
1	1	1	1	2	6	2	0	0	1
0	2	2	0	3	4	3	0	0	1
1	2	1	0	2	5	3	0	0	1

When $\text{das}=16$ and activating $\text{dom}()$ for $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ with $e = 4$ and the 4 sets just found in $\text{das}=15$, we get 4 sets:

0	2	3	1	3	4	2	0	0	1
1	2	2	1	2	5	2	0	0	1
0	3	3	0	3	3	3	0	0	1
1	3	2	0	2	4	3	0	0	1

Executing ASG with $\text{das}=17$ and activating $\text{dom}()$ for $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, the 4 sets found in $\text{das}=15$ and the 4 sets in $\text{das}=16$, we get 9 sets all with pattern: 1 3 3 1 2 4 2 0 0 1

When $\text{das} \geq 18$ and activating $\text{dom}()$ for $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, the 4 sets found in $\text{das}=15$, the 4 sets in $\text{das}=16$ and the pattern of $\text{das}=17$ we get only patterns which are dominated.

The 9 patterns we found in $\text{das}=15, 16, 17$ are dominated by φ_2 (except the first one which is dominated by φ_1); use domination in:

00	01	02	03	10	11	12	20	21	30
						2	0	0	1
						1	1	1	0

We have: $pq^2 - p^2r - p^2q + p^3 = p[(p-q)^2 + p(q-r)] \geq 0$.

D 75 probability classes; $p=0.71$, $q=0.23$, $r=0.06$

No.	adequate set:													prob.	order	case
1	4	5	7	8	9	13	14	16	17	18	20	24	1	0.230355	3	
2	1	2	8	12	13	15	16	20	21	22	24	25		0.230355		
3	3	6	8	10	11	13	14	19	20	22	23	24		0.230355		
4	3	5	6	10	11	16	17	19	20	21	25	26	2	0.230542	3	
5	4	5	7	8	9	11	15	18	22	23	25	26		0.230542		
6	1	2	7	12	14	15	17	19	21	23	24	26		0.230542		
7	3	5	6	8	10	11	16	19	21	23	24	26	3	0.23227	3	
8	4	5	7	9	11	15	17	18	20	22	24	26		0.23227		
9	1	2	7	8	12	14	15	19	20	21	25	26		0.23227		
10	4	5	7	8	9	13	14	16	17	18	19	24	4	0.237597	6	
11	4	5	7	8	9	13	14	16	17	18	20	21		0.237597		
12	1	2	5	12	13	15	16	20	21	22	24	25		0.237597		
13	1	2	8	11	12	13	15	16	21	22	24	25		0.237597		
14	3	6	7	10	11	13	14	19	20	22	23	24		0.237597		
15	3	6	8	10	11	13	14	15	19	20	22	23		0.237597		
~	~	~	~	~	~	~	~	~	~	~	~	~	~	~		
~	~	~	~	~	~	~	~	~	~	~	~	~	~	~		
70	1	2	4	5	6	12	15	17	19	20	22	23	17	0.288333	12	555
71	2	3	4	6	7	10	11	17	21	22	24	25		0.288333		555
72	2	3	5	6	10	12	15	16	19	20	23	25		0.288333		663
73	1	2	4	5	12	14	15	17	18	19	24	25		0.288333		663
74	1	2	6	7	10	11	12	14	21	23	24	25		0.288333		663
75	1	2	7	8	10	11	12	16	17	18	21	23		0.288333		555
76	2	4	5	9	10	15	16	18	19	23	24	25		0.288333		555
77	2	4	7	8	9	11	12	17	18	21	22	25		0.288333		663
78	3	4	6	7	10	11	16	17	18	20	21	23		0.288333		663
79	3	5	6	8	10	12	14	15	17	18	19	25		0.288333		555
80	4	5	6	8	9	10	15	17	18	19	22	23		0.288333		663
81	4	6	7	9	11	12	14	18	20	21	23	25		0.288333		555
~	~	~	~	~	~	~	~	~	~	~	~	~	~	~		
~	~	~	~	~	~	~	~	~	~	~	~	~	~	~		
324	0	1	3	4	8	9	10	12	13	20	24	26	75	0.838468	1	

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