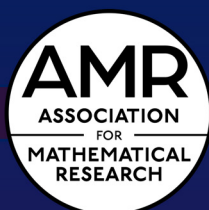
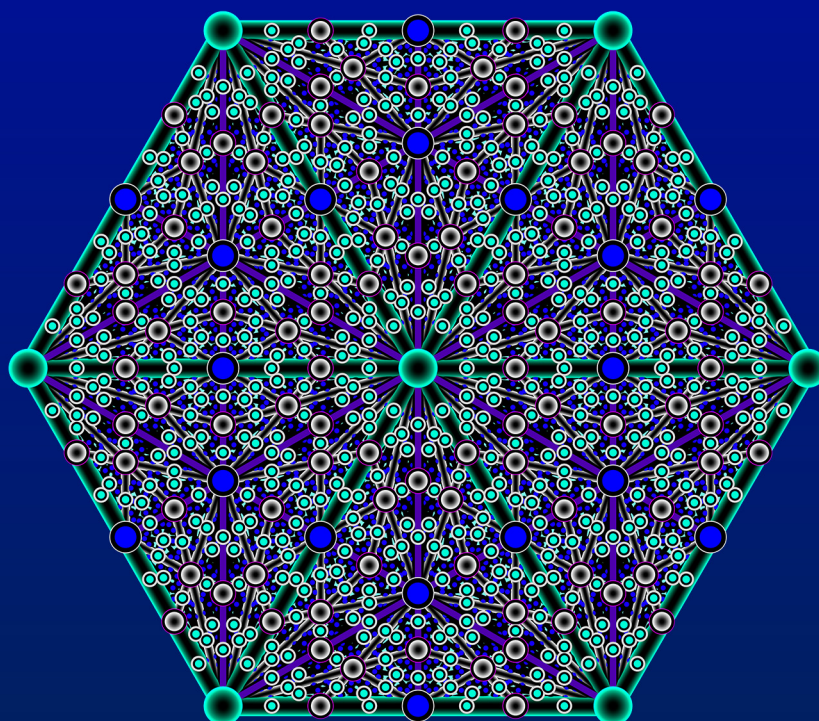


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Greenberg’s conjecture for real quadratic number fields

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Abstract:

We compute the 3-class groups A_n of the fields F_n in the cyclotomic $\mathbf{Z}_3$ -extensions of the real quadratic fields of discriminant $f < 100,000$. In all cases the orders of A_n remain bounded as n goes to infinity. This is in agreement with Greenberg’s conjecture.

Key words and phrases:

Iwasawa theory; Greenberg’s conjecture; Real number fields; Algebraic number theory

1 Introduction

Let F be a totally real number field and let p be a prime. Let

$$F = F_0 \subset F_1 \subset F_2 \subset \dots$$

denote the cyclotomic $\mathbf{Z}_p$ -extension of F . The p -class group A_n is the p -part of the ideal class group of the ring of integers of F_n . In his 1971 thesis, Ralph Greenberg conjectured that $\#A_n$ remains bounded as $n \rightarrow \infty$. See [Gre71, Gre76] and [Gre01, Conjecture (3.4)]. This is the so-called “ $\lambda = 0$ ”-conjecture of Iwasawa theory. In this note we report on a computation for the prime $p = 3$ involving the 30394 real quadratic fields $\mathbf{Q}(\sqrt{f})$ of discriminant $f < 100,000$. See [FK86, IS97, IS96, KS95] for earlier computations. As a consequence we obtain the following result.

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Theorem 1.1. *Greenberg’s conjecture is true for $p = 3$ and the real quadratic fields of discriminant $f < 100,000$.*

For each of the real quadratic fields with discriminant f in the range of our computation we have computed a certain Galois module $C(f)$, the finiteness of which is equivalent to Greenberg’s conjecture. In this introduction we define the module $C(f)$. In the subsequent sections we explain our computation and its results. See [KS95] for the algebraic properties of $C(f)$ in the case $f \not\equiv 1 \pmod{3}$. The slightly different case $f \equiv 1 \pmod{3}$ is discussed in detail in [Pao02] and in Section 4 of this paper.

Let $F = \mathbf{Q}(\sqrt{f})$ be a real quadratic field of discriminant f . The Galois module $C(f)$ is defined in terms of cyclotomic units as follows. For $k \geq 1$ let ζ_k denote a primitive k -th root of unity. For $n \geq 0$ the n -th layer in the cyclotomic $\mathbf{Z}_3$ -extension of F is

$$F_n = \mathbf{Q}(\sqrt{f}, \zeta_{3^{n+1}} + \zeta_{3^{n+1}}^{-1}).$$

The field F_n is a subfield of the cyclotomic field $\mathbf{Q}(\zeta_{3^{n+1}f})$. It is a cyclic degree 3^n extension of $F_0 = \mathbf{Q}(\sqrt{f})$. Its ring of integers O_n contains cyclotomic units. See [Sin80, Section 4]. The 3-part of the quotient of the unit group O_n^* by the subgroup generated by the cyclotomic units is a finite group denoted by B_n . It is known that the groups A_n and B_n have the same cardinality [Sin80, Theorems 4.1 and 5.3]. Therefore Greenberg’s conjecture is true for the field F if and only if $\#B_n$ remains bounded as $n \rightarrow \infty$.

When the discriminant f is not congruent to 1 $\pmod{3}$, we let C_n denote the dual of the group B_n for $n \geq 0$. When $f \equiv 1 \pmod{3}$, we let C_n denote the dual of the group $\tilde{B}_n$. Here $\tilde{B}_n$ sits in an exact sequence of the form

$$0 \longrightarrow \tilde{B}_n \longrightarrow B_n \xrightarrow{\phi_n} \mathbf{Z}_3 / \log_3 \eta_0 \mathbf{Z}_3.$$

where for $\varepsilon \in O_n^*$ we put $\phi_n(\varepsilon) = \frac{1}{3^n} \log_3(N_n(\varepsilon))$. Here $N_n: F_n^* \rightarrow \mathbf{Q}(\sqrt{f})^*$ is the norm map. Since the 3-adic logarithm of a generator η_0 of the group of cyclotomic units in $\mathbf{Q}(\sqrt{f})$ is not zero, the rightmost group is a finite cyclic group. It follows that $[B_n : \tilde{B}_n]$ and hence the quotient $\#B_n / \#C_n$ is bounded independently of n . Therefore Greenberg’s conjecture is true if and only if $\#C_n$ remains bounded as $n \rightarrow \infty$.

By [KS95, Lemma 2.1] and Section 4, the natural maps $B_m \rightarrow B_n$ are injective and the natural maps $C_n \rightarrow C_m$ are surjective for $n \geq m$. Let $C(f)$ denote the projective limit of the C_n . Then $C(f)$ is a Galois module and hence in the usual way a module over the Iwasawa algebra $\Lambda \cong \mathbf{Z}_3[[T]]$. It follows from the structure of the cyclotomic units that $C(f)$ is a cyclic Λ -module. See [KS95, Theorem 2.4] and Section 4. In other words, we have

$$C(f) = \varprojlim C_n \cong \Lambda/J, \quad \text{for some ideal } J \subset \Lambda.$$

The vanishing of the Iwasawa μ -invariant of $\mathbf{Q}(\sqrt{f})$ means that J contains a monic polynomial and hence that $C(f)$ is a finitely generated $\mathbf{Z}_3$ -module. See [FW79]. Greenberg’s conjecture affirms that $C(f)$ is actually *finite*, so that $C_n = C(f)$ for sufficiently large n .

We have computed the Galois modules $C(f)$ for $f < 100,000$. It took about two weeks on a workstation with Intel processor i5. We found the following, which is equivalent to Theorem 1.1.

Theorem 1.2. *For $p = 3$ and for all discriminants $f < 100,000$ the module $C(f)$ is finite.*

In most cases the module $C(f)$ is actually zero. Indeed, for only 3359 out of the 30394 real quadratic fields considered, $C(f)$ is not zero and, equivalently, J is a proper Λ -ideal. This is about 11% of all cases. Of these, 2118 have J equal to the maximal ideal $(3, T)$ of Λ . In these cases $C(f)$ has order 3. For the remaining 1241 discriminants the module $C(f)$ is strictly larger. This is approximately 4% of all cases.

Rather than listing each ideal J , we indicate in Sections 3 and 5 how often ideals of a certain type appear in our computation. The full list of ideals may be of interest in itself and is available on GitHub [Iwa]. In Sections 3 and 5 we also single out some discriminants for which the ideal J has a remarkable shape.

2 The case $f \not\equiv 1 \pmod{3}$

In this section we give a brief description of the algorithm in the case where the discriminant f is congruent to 0 or 2 modulo 3. This case is discussed in detail in [KS95]. Let $F = \mathbf{Q}(\sqrt{f})$ be a real quadratic field of discriminant f . Put $f' = f/3$ when $f \equiv 0 \pmod{3}$ and $f' = f$ when $f \equiv 2 \pmod{3}$. For $n \geq 0$ the n -th layer F_n in the cyclotomic tower of F is a subfield of $\mathbf{Q}(\zeta_{3^{n+1}f'})$. The cyclotomic unit $1 - \zeta_{3^{n+1}f'}$ is contained in $\mathbf{Q}(\zeta_{3^{n+1}f'})$. Put

$$\eta_n = \text{Norm}_{\mathbf{Q}(\zeta_{3^{n+1}f'})/F_n} (1 - \zeta_{3^{n+1}f'})^{\sigma-1}.$$

Here σ is the non-trivial automorphism in $\text{Gal}(F_n/\mathbf{Q}_n) \cong \text{Gal}(\mathbf{Q}(\sqrt{f})/\mathbf{Q})$.

In [KS95] it is explained that the Galois module generated by η_n is free of rank 1 over $\mathbf{Z}[G_n]$. Here G_n denotes $\text{Gal}(F_n/F_0)$. This implies that the Galois module $C(f)$ described in the introduction is a *cyclic* module over the Iwasawa algebra $\Lambda = \varprojlim \mathbf{Z}_3[G_n] \cong \mathbf{Z}_3[[T]]$. So we have $C(f) = \Lambda/J$ for some Λ -ideal J . For $n \geq 0$ we put $\omega_n(T) = (1+T)^{p^n} - 1$ and we write (ω_n) for the Λ -ideal generated by it. In [KS95] it is explained that in this case we have

$$C_n = C(f)/\omega_n C(f) = \Lambda/(J + (\omega_n)), \quad \text{for all } n \geq 0.$$

The Galois module $C(f)$ is finite if and only if $\omega_n C(f) = 0$ and hence $C(f) = C_n$ for some $n \geq 0$. By Nakayama's lemma this happens if and only if $J + (\omega_n) = J + (\omega_{n+1})$ for some $n \geq 0$. This observation leads to the following algorithm. For $n = 0, 1, 2, \dots$, we compute the shrinking ideals $J + (\omega_n)$ until we find that $J + (\omega_n) = J + (\omega_{n+1})$.

Our method for computing the ideals $J + (\omega_n)$ runs as follows. For a given n we first calculate a lot of elements in the ideal. As is explained in [KS95], this involves calculations with cyclotomic units modulo primes $r \equiv 1 \pmod{f'3^{n'}}$ for suitable $n' > n$. This leads to an *upper bound* for $\Lambda/(J + (\omega_n))$. To obtain a *lower bound* we employ a method due to G. and M.-N. Gras [GG77]. This involves calculations with high precision approximations of the cyclotomic units in $F_n \otimes \mathbf{R}$. See also [KS95, Section 4]. Clearly, when the upper and lower bounds agree, we have determined $J + (\omega_n)$ and hence $C_n = \Lambda/(J + (\omega_n))$.

The calculation of the lower bound for C_n becomes very time consuming and takes a lot of memory as n grows. This is caused by the high precision computations with units in cyclotomic fields of seven digit conductors and degrees in the hundreds. In fact, for most discriminants f it becomes infeasible when n exceeds 2. Fortunately, for most f we find that $J + (\omega_n) = J + (\omega_{n+1})$ and hence $C(f) = C_n$ for $n \leq 2$.

In the rare cases where we need to consider $J + (\omega_n)$ for $n \geq 3$, it is still feasible to compute the upper bound in the sense that we can easily calculate a lot of elements in the ideal $J + (\omega_n)$. An application of the Chebotarev density theorem suggests that these elements probably *generate* $J + (\omega_n)$, so that our upper bound is actually *equal* to the lower bound, but we have no rigorous proof of this.

Fortunately, we can still rigorously prove that $C(f) = \Lambda/J$ is finite and thus confirm Greenberg's conjecture even when we cannot use our algorithm to compute lower bounds for $\Lambda/(J + (\omega_n))$. It suffices to have an upper bound for n and a lower bound for *some* $m \leq n$ to which the following lemma applies. In the range of our computations this always works out with $n \geq m = 2$.

Lemma 2.1. *Let M be a finitely generated Λ -module. Suppose that for certain integers $n \geq m \geq 0$ and $b \geq a \geq 0$ we have*

$$\#M/\omega_m M \geq p^a \quad \text{and} \quad \#M/\omega_n M \leq p^b.$$

If $b - a < n - m$, then $\omega_n M = 0$. In particular, if $M/\omega_n M$ is finite, so is M .

Proof. In the filtration

$$\omega_n M \subset \omega_{n-1} M \subset \dots \subset \omega_{m+1} M \subset \omega_m M$$

there are $n - m$ inclusions. We have inequalities

$$\#(\omega_m M / \omega_n M) = \frac{\#M/\omega_n M}{\#M/\omega_m M} \leq p^{b-a} < p^{n-m}.$$

It follows that one of the inclusions must be an equality. So we have $\omega_{k+1} M = \omega_k M$ for some $k = m, \dots, n - 1$. Then $x = \omega_{k+1}/\omega_k$ is an element of the maximal ideal of Λ that has the property that $x\omega_k M = \omega_k M$. Nakayama's lemma implies then $\omega_k M = 0$. It follows that $\omega_n M$ is zero, as required. $\square$

3 Numerical data for discriminants $f \not\equiv 1 \pmod{3}$

3.1 Case $f \equiv 0 \pmod{3}$

There are 7606 real quadratic fields with discriminant $f \equiv 0 \pmod{3}$ and $f < 100,000$. For precisely 769 of them the Galois module $C(f) = \Lambda/J$ is not zero. This is approximately 10%. For 513 discriminants J is equal to the maximal ideal $(3, T)$ of Λ . For the remaining 256 discriminants J is strictly smaller. Table 3.1 contains some data.

The rows of Table 3.1 correspond to the *level of stabilization* n . This means that n is the smallest integer for which the ideals $J + (\omega_n)$ and $J + (\omega_{n+1})$ are equal and hence $J = J + (\omega_n)$. In particular, we have $\Lambda/J = C(f) = C_n$. The number n is also the smallest for which $\omega_n = (1 + T)^{3^n} - 1$ is in J . Equivalently, 3^n is the order of $1 + T$ in the multiplicative group $(\Lambda/J)^*$.

The columns are indexed by the symbols T^k for $k = 1, 2, \dots$. The entry in the n -th row and the T^k -column is the number of discriminants for which the level of stabilization is n , and the image of J in the ring $\mathbf{F}_3[[T]]$ is the ideal (T^k) . Since ω_n is congruent to T^{3^n} modulo 3, the (n, T^k) -entry is zero whenever $k > 3^n$. In particular, in the row corresponding to $n = 0$, all entries with $k > 1$ are zero.

n	T	T^2	T^3	Total
0	536	0	0	536
1	112	50	2	164
2	35	7	2	44
3	15*	0	0	15
4	5*	1*	0	6
5	2*	0	0	2
6	2*	0	0	2
	707	58	4	769

 Table 3.1: The modules Λ/J for $f \equiv 0 \pmod{3}$.

In the first column we count the discriminants for which the ideal J is of the form $J = (T - a, b)$ for certain $a, b \in \mathbf{Z}$. For 536 discriminants we have $a = 0$ and there is stabilization at level $n = 0$. This means that $\#C_0 = \#C_1$ or, equivalently $\#A_0 = \#A_1$. The discriminants for which J is equal to the maximal ideal of Λ are included here. This entry was checked by computing the class numbers of the fields F_0 and F_1 of degrees 2 and 6 respectively using a few lines of PARI/GP [The20] code. For the other entries in the first column, we have $a \notin b\mathbf{Z}_3$ and stabilization occurs at level $n = v_3(b/a)$.

An asterisk indicates that we do not have a rigorous lower bound for $C(f)$ for some of the discriminants appearing in this entry. However, our upper bound is very likely to be sharp, so that almost certainly $C(f)$ is isomorphic to Λ/J . In each case Lemma 2.1 was applied to prove Greenberg's conjecture. The 62 cases appearing in the second and third columns were dealt with using the polynomial arithmetic of Magma [BCP97]. We single out nine discriminants f for special mention.

f	J	n	T^k
31989	$(T - 996, 2187)$	6	T
38424	$(T + 261, 2187)$	5	T
59061	$(T^2 + 3T - 9, 81)$	4	T^2
60513	$(T^3 + 3, 3T, 9)$	2	T^3
61629	$(T^3, 3)$	1	T^3
69117	$(T + 69, 729)$	5	T
71049	$(T^3, 3)$	1	T^3
76584	$(T^3 + 3, 3T, 9)$	2	T^3
95385	$(T - 2988, 6561)$	6	T

 Table 3.2: Exotic Galois modules for $f \equiv 0 \pmod{3}$.

3.2 Case $f \equiv 2 \pmod{3}$

There are 11394 real quadratic fields with discriminant $f \equiv 2 \pmod{3}$ and $f < 100,000$. For precisely 1250 of them the Galois module $C(f) = \Lambda/J$ is not zero. This is approximately 11% of all discriminants. For 781 discriminants J is equal to the maximal ideal $(3, T)$ of Λ . For the remaining 469 discriminants J is strictly smaller. This is about 4% of all cases. Table 3.3 contains some data.

n	T	T^2	T^3	T^4	Total
0	827	0	0	0	827
1	158	87	8	0	253
2	101	7	4	1	113
3	36*	2*	0	0	38
4	13*	1*	0	0	14
5	4*	0	0	0	4
6	1*	0	0	0	1
	1140	97	12	1	1250

Table 3.3: The modules Λ/J for $f \equiv 2 \pmod{3}$.

The interpretation of the entries of the table is the same as in the case $f \equiv 0 \pmod{3}$. The 781 discriminants with $J = (3, T)$ are included in the entry with $n = 0$ of the first column. Also in this case the discriminants in the first column were taken checked using a few lines of PARI/GP code. The other 110 cases were dealt with using the polynomial arithmetic of Magma. We single out nine discriminants for special mention.

f	J	n	T^k
14165	$(T - 255, 729)$	5	T
16673	$(T + 462, 2187)$	6	T
29165	$(T - 282, 729)$	5	T
47633	$(T^2 - 9, 3T - 90, 243)$	4	T^2
51809	$(T^2 + 18, 3T - 18, 81)$	3	T^2
71921	$(T^2 + 18, 3T + 18, 81)$	3	T^2
76604	$(T + 294, 729)$	5	T
90005	$(T + 15, 729)$	5	T
98105	$(T^4 + 3, 3T, 9)$	2	T^4

Table 3.4: Exotic Galois modules for $f \equiv 2 \pmod{3}$.

4 The case $f \equiv 1 \pmod{3}$

As before we write $F = \mathbf{Q}(\sqrt{f})$ and F_n for the n -th layer in the cyclotomic $\mathbf{Z}_3$ -extension of $F = F_0$. When the discriminant f is congruent to 1 (mod 3), our method to compute the Galois module $C(f)$ is the same, but the details are slightly different. See [Pao02]. The differences are caused by the fact that the Galois module generated by the cyclotomic unit η_n is *not free* over the ring $\mathbf{Z}[G_n]$ when $f \equiv 1 \pmod{3}$. Here η_n is defined in Section 2 and G_n denotes $\text{Gal}(F_n/F_0)$. Indeed, in this case we have $N_n\eta_n = 1$, where N_n is the norm map $F_n^* \rightarrow F^*$. When $f \equiv 1 \pmod{3}$, the Galois module Cyc_n of cyclotomic units in F_n on which σ acts as -1 , is a direct product of the submodules generated by η_n and η_0 . Here η_0 is the cyclotomic unit in F_0 . It generates a group isomorphic to $\mathbf{Z}$ with trivial Galois action. On the other hand, the Galois module $\langle \eta_n \rangle$ generated by η_n is free of rank 1 over the ring $\mathbf{Z}[G_n]/(N_n)$. See [Sin80].

The submodule $\tilde{B}_n$ of B_n that was defined in the introduction, is isomorphic to $O_{n,1}^*/\langle \eta_n \rangle$. Here $O_{n,1}^*$ denotes the part of the kernel of the norm map $N_n: O_n^* \rightarrow O_0^*$ on which σ acts as -1 . The Galois modules $O_{n,1}^*$, $\langle \eta_n \rangle$ and $\tilde{B}_n$ are killed by the norm N_n and are hence $\mathbf{Z}[G_n]/(N_n)$ -modules. Since $\langle \eta_n \rangle$ is free of rank 1, it is more convenient to deal with $\tilde{B}_n$ rather than with B_n itself. For instance, from the exactness of the sequence of $\mathbf{Z}[G_n]/(N_n)$ -modules

$$0 \longrightarrow \langle \eta_n \rangle \longrightarrow O_{n,1}^* \longrightarrow \tilde{B}_n \longrightarrow 0$$

one deduces that the natural map $\tilde{B}_m \rightarrow \tilde{B}_n$ identifies $\tilde{B}_m$ with the kernel of the endomorphism ω'_m of $\tilde{B}_n$ for $m \leq n$. Here we put $\omega'_m = \omega_m/T$. In particular, we have $\omega'_0 = 1$ and $C_0 = 0$. It follows that the Galois module $C(f)$ is isomorphic to Λ/J for some ideal J and $C_n = C(f)/\omega'_n C(f) = \Lambda/(J + (\omega'_n))$ for all $n \geq 0$.

Our strategy is the one explained in Section 2: for each $n = 1, 2, \dots$, we compute the shrinking ideals $J + (\omega'_n)$ until we find $J + (\omega'_n) = J + (\omega'_{n+1})$, in which case Nakayama's lemma implies that $J = J + (\omega'_n)$ and hence $C(f) = C_n$ and we are done. When $f \equiv 1 \pmod{3}$ the issues with upper bounds and lower bounds are similar to the ones described in Section 2 for $f \not\equiv 1 \pmod{3}$. In particular, we can still prove that $C(f) = \Lambda/J$ is finite in each case in the range of our computations. When the lower bound is not available for some $n \geq 3$, we invoke Lemma 2.1 with ω_m and ω_n replaced by ω'_m and ω'_n respectively.

It is not relevant for our algorithm and computations, but in the rest of this section we analyze the cokernel of the inclusion map $\tilde{B}_n \hookrightarrow B_n$. For $n \geq 0$, let U_n denote the part of the unit group $(O_n \otimes \mathbf{Z}_3)^*$ on which σ acts as -1 . Since $\tilde{B}_n$ is the kernel of the map $B_n \rightarrow U_0/\langle \eta_0 \rangle$ induced by $\varepsilon \mapsto \sqrt[n]{N_n} \varepsilon$ for $\varepsilon \in O_n^*$, the quotient $B_n/\tilde{B}_n$ is isomorphic to a subgroup of the cyclic group $U_0/\langle \eta_0 \rangle$ and is hence bounded independently of n . This can be made more precise.

The group $N_n U_n$ is equal to the subgroup $U_0^{p^n}$ of U_0 . It follows that $N_n O_n^*$ is contained in $U_0^{p^n}$. Put

$$\sqrt[p^n]{N_n O_n^*} = \{u \in U_0 : u^{p^n} \in N_n O_n^*\}.$$

For every $n \geq 0$ we have inclusions

$$N_n O_n^{*p} = N_{n+1} O_n^* \subset N_{n+1} O_{n+1}^* \subset N_n O_n^*.$$

It follows that we have a filtration

$$O_0^* \subset \dots \subset \sqrt[p^n]{N_n O_n^*} \subset \sqrt[p^{n+1}]{N_{n+1} O_{n+1}^*} \subset \dots \subset U_0,$$

with successive subquotients of order 1 or p . The fact that $N_n \text{Cyc}_n$ is equal to $\langle \eta_0^{p^n} \rangle$ gives rise to the isomorphisms

$$B_n/\tilde{B}_n \cong N_n O_n^* / \langle \eta_0^{p^n} \rangle \cong \sqrt[p^n]{N_n O_n^*} / \langle \eta_0 \rangle.$$

This leads to the following filtration

$$O_0^* / \langle \eta_0 \rangle \subset \dots \subset B_n / \tilde{B}_n \subset B_{n+1} / \tilde{B}_{n+1} \subset \dots \subset U_0 / \langle \eta_0 \rangle.$$

with successive subquotients of order 1 or p . The leftmost group is cyclic of order $h_0 = \#A_0$ and the rightmost group has order $\log_3 \eta_0$. Writing ε_0 for a fundamental unit of $F = \mathbf{Q}(\sqrt{f})$, there are $v_3 \log_3 \varepsilon_0$ distinct steps in this filtration. By Nuccio [Nuc10] we have $B_n / \tilde{B}_n = U_0 / \langle \eta_0 \rangle$ when n is sufficiently large.

5 Numerical data for discriminants $f \equiv 1 \pmod{3}$.

There are 11394 real quadratic fields with discriminant $f \equiv 1 \pmod{3}$ and $f < 100,000$. For precisely 1340 of them the module $C(f)$ is not zero. This is approximately 12% of all discriminants. For 824 discriminants J is equal to the maximal ideal $(3, T)$ of Λ . For the remaining 516 discriminants the ideal J is strictly smaller. This is 4.5% of all cases.

The mathematics is a bit different in this case. First of all, the groups A_0, B_0 are irrelevant for our computations and we have $C_0 = 0$. In addition, every module C_n is a cyclic module over the ring $\Lambda/(\omega_n)$ that is killed by ω'_n . In particular, C_1 is a cyclic module over the discrete valuation ring $\Lambda/(\omega'_1)$, where $\omega'_1 = \omega_1/T = T^2 + 3T + 3$. Since T is a uniformizer of the ring $\Lambda/(\omega'_1)$, the module C_1 is isomorphic to $\Lambda/(T^2 + 3T + 3, T^k)$ for some $k \geq 0$.

n	T	T^2	T^3	T^4	T^5	Total
1	824	79	0	0	0	903
2	249	18	8	1	0	276
3	88	7	1	0	1	97
4	47*	3*	0	0	0	50
5	9*	0	1*	0	0	10
6	2*	0	0	0	0	2
7	2*	0	0	0	0	2
	1221	107	10	1	1	1340

Table 5.1: The modules Λ/J for $f \equiv 1 \pmod{3}$.

By Nakayama's lemma the ideal J contains a monic polynomial of degree 1 if and only if the ideal $(T^2 + 3T + 3, T^k)$ does. If J is a proper ideal, this happens precisely when $k = 1$, in which case C_1 is isomorphic to the order 3 module $\Lambda/(3, T)$. These cases appear in the first column of Table 5.1 and were computed using PARI/GP. Their ideals J are of the form $(T - a, b)$ with level of stabilization equal to $v_3(b)$. In particular, the first entry contains the 824 discriminants for which J is equal to the ideal $(3, T)$.

The 119 entries in the remaining columns of Table 5.1 were taken care of using Magma's polynomial arithmetic.

We single out eleven discriminants for special mention.

f	J	n	T^k
15217	$(T^4 + 3, 3T, 9)$	2	T^4
30904	$(T^3 - 27, 3T - 63, 243)$	5	T^3
39256	$(T + 621, 2187)$	7	T
40441	$(T^2, 9T - 27, 81)$	4	T^2
44053	$(T + 348, 729)$	6	T
57832	$(T^2 + 27, 3T - 27, 81)$	4	T^2
71821	$(T^3 + 18, 3T + 9, 27)$	3	T^3
78037	$(T - 849, 2187)$	7	T
80056	$(T^5 + 9T + 9, 3T^2 + 18, 27)$	3	T^5
81769	$(T^2 + 18, 3T + 9, 81)$	4	T^2
96712	$(T - 30, 729)$	6	T

Table 5.2: Exotic Galois modules for $f \equiv 1 \pmod{3}$.

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