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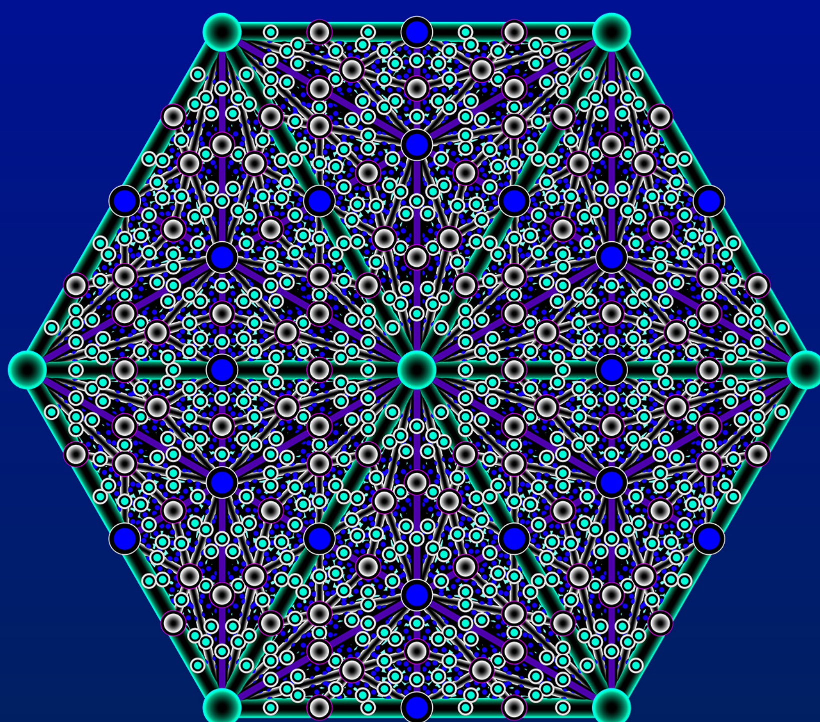

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In Search of Approximate Polynomial Dependencies Among the Derivatives of the Alternating Zeta Function

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Abstract:
It is well-known that the Riemann zeta function does not satisfy any *exact* polynomial differential equation. Here we present numerical evidence suggesting the existence of *approximate* polynomial dependencies between the values of the alternating zeta function and its initial derivatives.
A number of conjectures is stated.

Key words and phrases:
Dirichlet eta function; Approximate polynomial relations among the derivatives

1 Introduction

The *Riemann zeta function* can be defined via a *Dirichlet series*, namely,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.1}$$

The series converges for $\Re(s) > 1$ but the zeta function can be analytically extended to the whole complex plane with the exception of the simple pole at $s = 1$.

The zeta function is one of the most mysterious mathematical objects. The celebrated *Riemann Hypothesis* has remained unproved for more than a century and a half.

The study of the zeta function would be easier if we could find a suitable differential equation satisfied by this function. However, this is impossible. D. Hilbert [Hil00] asserted that the zeta function does

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not satisfy any algebraic differential equation. A proof was given later by V. E. E. Stadigh [Sta02] and, in a more general setting, by D. D. Mordukhai-Boltovskoi [MB14] (see also [Ost20, MB32]). These "negative" results were further extended to broader classes of differential equations (see, for example, recent publication [CW21] and references there).

For real values of the argument, the zeta function was studied already by L. Euler. Likewise, he considered the *alternating zeta function* (known also as *Dirichlet eta function*)

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}. \tag{1.2}$$

The two functions, (1.1) and (1.2), are interconnected:

$$\eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s). \tag{1.3}$$

The factor $1 - 2 \cdot 2^{-s}$ vanishes at $s = 1$, and thus $\eta(s)$ is an entire function. Because of this, it is often easier to work with $\eta(s)$. Moreover, the series (1.2) converges in a larger half-plane $\Re(s) > 0$.

The above mentioned "negative" results for the zeta function can be transferred to the eta function. In particular, the eta function cannot satisfy any *exact* polynomial differential equation. In this paper we demonstrate *approximate* polynomial relations among the values of the alternating zeta function and its derivatives. This relationship was found experimentally, and so far no "explanation" has been given for these phenomena.

More precisely, we define several series of pairs of polynomials with integer coefficients

$$\langle V_N(y_0, \dots, y_k, \dots, y_{N-1}),\, W_N(y_0, \dots, y_k, \dots, y_{N-1})\rangle, \quad N = 1, 2, \dots\,. \tag{1.4}$$

For each such series, we state a conjecture that

$$\frac{V_N(\eta(a), \dots, \eta^{\langle n \rangle}(a), \dots, \eta^{\langle N-1 \rangle}(a))}{W_N(\eta(a), \dots, \eta^{\langle n \rangle}(a), \dots, \eta^{\langle N-1 \rangle}(a))} \xrightarrow[N \to \infty]{} 1 \tag{1.5}$$

except for countably many values of a. To support such conjectures we demonstrate certain numerical data. They show that already for relatively small values of N the left-hand side in (1.5) can be very close to 1.

The paper consists of two parts. In Part I we present raw numerical data and discuss their peculiarities. Based on our observations, in Part II we construct polynomials (1.4) with the expected property (1.5). In the Concluding remark we briefly discuss possible extension of our observation and conjectures to the Riemann zeta function.

Preliminary versions of this paper [Mat24a] and [Mat24b] contain more numerical data and conjectures.

Part I
Numerical observations

2 An unreachable goal

Let us start by considering a very ambitious and therefore unattainable goal: *for a particular positive integer N, find a linear polynomial*

$$P(y_0, y_1, \ldots, y_{N-1}) = y_0 + c_1 y_1 + \cdots + c_{N-1} y_{N-1} - b \tag{2.1}$$

with numerical coefficients

$$b, c_1, \ldots, c_{N-1} \tag{2.2}$$

such that for all a

$$P\big(\eta(a), \eta'(a), \ldots, \eta^{\langle N-1\rangle}(a)\big) = 0. \tag{2.3}$$

If such a polynomial P existed, we could find its coefficients (2.2) in the following way. First, we select a set $\mathfrak{A} = \{a_0, \ldots, a_{N-1}\}$ of N complex numbers in general position. Then, we solve the linear system

$$\eta(a_j) + c_1(\mathfrak{A})\eta'(a_j) + \cdots + c_{N-1}(\mathfrak{A})\eta^{\langle N-1\rangle}(a_j) = b(\mathfrak{A}), \quad j = 0, \ldots, N-1 \tag{2.4}$$

in unknowns

$$b(\mathfrak{A}),\ c_1(\mathfrak{A}),\ \ldots,\ c_{N-1}(\mathfrak{A}). \tag{2.5}$$

Our goal would be achieved if all numbers (2.5) were the same for all $\mathfrak{A}$. As explained in the introduction, this is not the case.

Let us examine numerical data for two particular sets depicted in Fig. 2.1. These sets are specimens of *grids* and *discrete circles* which in general case are defined as follows. For complex a, δ_1, δ_2 and non-negative integers N_1, N_2

$$\mathfrak{A}_{\mathrm{G}}(a, \delta_1, \delta_2, N_1, N_2) = \{a + k\delta_1 + l\delta_2 \mid k = 0, \ldots, N_1, l = 0, \ldots, N_2\}, \tag{2.6}$$

and for complex c, r and positive integer N

$$\mathfrak{A}_{\mathrm{C}}(c, r, N) = \{c + \mathrm{e}^{2\pi\mathrm{i}k/N} r \mid k = 0, \ldots, N-1\}. \tag{2.7}$$

Table 2.1 presents the values of the c-coefficients from the solutions of the system (2.4) for the two sets from Fig. 2.1. We observe that while the sets are rather different, there is certain similarity between numbers $c_k(\mathfrak{A}_1)$ and $c_k(\mathfrak{A}_2)$. As for the b-coefficients, $b(\mathfrak{A}_1)$ and $b(\mathfrak{A}_2)$, they are almost equal:

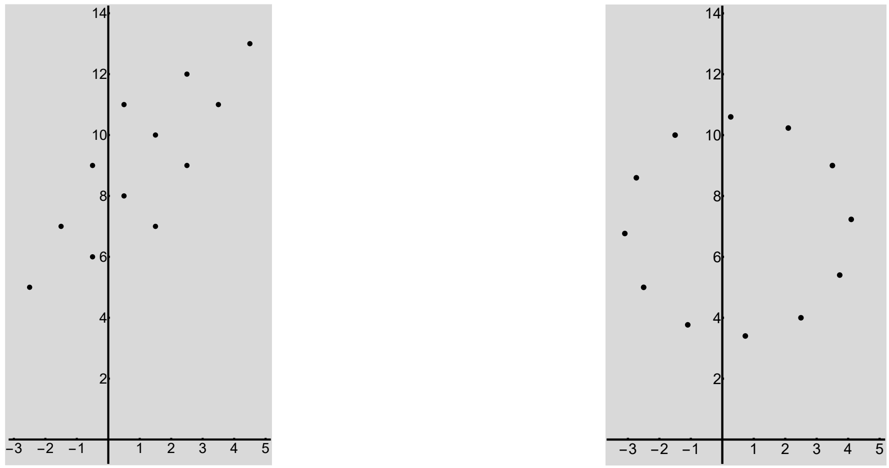
$$b(\mathfrak{A}_1) = 0.99999999999999430995\ldots + 0.00000000000000951329\ldots\mathrm{i}, \tag{2.8}$$

$$b(\mathfrak{A}_2) = 1.00000000000000637109\ldots + 0.00000000000003070321\ldots\mathrm{i}. \tag{2.9}$$

Moreover, Tables 2.2–2.3 show that $b(\mathfrak{A})$ from the solution of system (2.4) is surprisingly close to 1 for a selection of rather different grids and discrete circles:

$$b(\mathfrak{A}) \approx 1. \tag{2.10}$$

More relevant numerical data (for grids, discrete circles and “random” sets) are presented in [Mat24a, Tables 36–38]. We return to discussion of this phenomenon after introduction of some notation.



$\mathfrak{A}_1$ $\mathfrak{A}_2$

Figure 2.1: Grid $\mathfrak{A}_1 = \mathfrak{A}_{\mathrm{G}}(-2.5+5\mathrm{i}, 2+\mathrm{i}, 1+2\mathrm{i}, 2, 3)$ and discrete circle $\mathfrak{A}_2 = \mathfrak{A}_{\mathrm{C}}(0.5+7\mathrm{i}, 3+2\mathrm{i}, 12)$ defined respectively by (2.6) and (2.7).

3 Matrix notation

We rewrite system (2.4) in matrix notation and solve it by Cramer's rule. To this end we introduce the following notation.

Let D_N denote the determinant of the "canonical" $N \times N$ matrix:

$$D_N = \begin{vmatrix} x_{0,0} & \cdots & x_{0,N-1} \\ \vdots & \ddots & \vdots \\ x_{N-1,0} & \cdots & x_{N-1,N-1} \end{vmatrix}. \tag{3.1}$$

If E is some expression and m and n are particular integers such that $0 \le m < N$ and $0 \le n < N$, then

$$D_N(x_{m,n} \to E) \tag{3.2}$$

denotes the result of replacing $x_{m,n}$ by E in D_N.

Multiple replacements like

$$D_N(x_{1,1} \to E_1, x_{2,2} \to E_2) \tag{3.3}$$

are possible with the natural meaning.

Several similar replacements can be defined by a rule with parameters j and k. For example, $D_N(x_{1,k} \to E_k)$ is a short for

$$D_N(x_{1,0} \to E_0,\ \ldots,\ x_{1,N-1} \to E_{N-1}), \tag{3.4}$$

and $D_N(x_{j,k} \to E_{j,k})$ is a short for

$$\begin{split} D_N(x_{0,0} \to E_{0,0},\ \ldots,\ x_{0,N-1} \to E_{0,N-1}, \ldots, \\ x_{N-1,0} \to E_{N-1,0},\ \ldots,\ x_{N-1,N-1} \to E_{N-1,N-1}). \end{split} \tag{3.5}$$

	$\mathfrak{A} = \mathfrak{A}_1$	$\mathfrak{A} = \mathfrak{A}_2$
$c_1(\mathfrak{A})$	$7.30910\ldots - 0.26348\ldots\mathrm{i}$	$7.35147\ldots - 0.23197\ldots\mathrm{i}$
$c_2(\mathfrak{A})$	$23.85173\ldots - 1.81924\ldots\mathrm{i}$	$24.15410\ldots - 1.61238\ldots\mathrm{i}$
$c_3(\mathfrak{A})$	$45.96860\ldots - 5.55318\ldots\mathrm{i}$	$46.92441\ldots - 4.95964\ldots\mathrm{i}$
$c_4(\mathfrak{A})$	$58.22426\ldots - 9.89546\ldots\mathrm{i}$	$59.99059\ldots - 8.91557\ldots\mathrm{i}$
$c_5(\mathfrak{A})$	$50.94334\ldots - 11.42336\ldots\mathrm{i}$	$53.05986\ldots - 10.39502\ldots\mathrm{i}$
$c_6(\mathfrak{A})$	$31.43810\ldots - 8.94113\ldots\mathrm{i}$	$33.15830\ldots - 8.22831\ldots\mathrm{i}$
$c_7(\mathfrak{A})$	$13.68711\ldots - 4.81140\ldots\mathrm{i}$	$14.64820\ldots - 4.48445\ldots\mathrm{i}$
$c_8(\mathfrak{A})$	$4.11913\ldots - 1.75935\ldots\mathrm{i}$	$4.48375\ldots - 1.66349\ldots\mathrm{i}$
$c_9(\mathfrak{A})$	$0.81560\ldots - 0.41866\ldots\mathrm{i}$	$0.90549\ldots - 0.40230\ldots\mathrm{i}$
$c_{10}(\mathfrak{A})$	$0.09551\ldots - 0.05857\ldots\mathrm{i}$	$0.10851\ldots - 0.05731\ldots\mathrm{i}$
$c_{11}(\mathfrak{A})$	$0.00500\ldots - 0.00365\ldots\mathrm{i}$	$0.00583\ldots - 0.00365\ldots\mathrm{i}$

Table 2.1: Values of the c-coefficients from (2.4) for sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ depicted in Fig. 2.1.

a	δ_1	δ_2	N_1	N_2	N	$\lvert b(\mathfrak{A}) - 1\rvert$
$-200+15\mathrm{i}$	50	$50\mathrm{i}$	5	5	36	$1.0523\ldots \cdot 10^{-57}$
$-5+15\mathrm{i}$	5	$5\mathrm{i}$	5	5	36	$5.7251\ldots \cdot 10^{-64}$
$30\mathrm{i}$	0.0001	$0.0001\mathrm{i}$	5	5	36	$1.1424\ldots \cdot 10^{-53}$
$0.25+30\mathrm{i}$	0.0001	0	59	0	60	$5.3809\ldots \cdot 10^{-98}$
$0.5+30\mathrm{i}$	0	$3\mathrm{i}$	0	59	60	$2.0739\ldots \cdot 10^{-56}$
$-200+10000\mathrm{i}$	50	$50\mathrm{i}$	5	5	36	$3.8883\ldots \cdot 10^{-51}$

Table 2.2: Instances of accuracy of (2.10) for grids; $N = (N_1+1)(N_2+1)$ is the cardinality of the set $\mathfrak{A} = \mathfrak{A}_{\mathrm{G}}(a, \delta_1, \delta_2, N_1, N_2)$.

Replacement rules are applied consecutively from left to right, for example, in

$$D_N(x_{0,k} \to E_k,\ x_{j,0} \to F_j) \tag{3.6}$$

$x_{0,0}$ is replaced by E_0.

A replacement rule can be conditional, for example,

$$C_N = D_N(j = k \Rightarrow x_{j,k} \to x_{j,k} - \lambda) \tag{3.7}$$

is the characteristic polynomial of the "canonical" matrix from (3.1).

Similar to (3.2)–(3.5), replacement rules can be applied to polynomials C_N.

In the above notation $b(\mathfrak{A})$ from the solution of system (2.4) can be expressed as

$$b(\mathfrak{A}) = Q(\mathfrak{A}) \tag{3.8}$$

where

$$Q(\{a_0, \ldots, a_{N-1}\}) = \frac{D_N\left(x_{j,k} \to \eta^{\langle k \rangle}(a_j)\right)}{D_N\left(x_{j,0} \to 1, x_{j,k} \to \eta^{\langle k \rangle}(a_j)\right)}. \tag{3.9}$$

c	r	N	$\lvert b(\mathfrak{A}_{\mathrm{C}}(a,r,N))-1\rvert$
$10\mathrm{i}$	10^{-5}	24	$1.2861\ldots\cdot 10^{-32}$
$-1+20\mathrm{i}$	10^{-3}	36	$6.2669\ldots\cdot 10^{-52}$
$0.25+30\mathrm{i}$	10^{-1}	60	$5.4442\ldots\cdot 10^{-98}$
$0.6+40\mathrm{i}$	1	72	$1.1660\ldots\cdot 10^{-121}$
$1+50\mathrm{i}$	10	80	$4.6580\ldots\cdot 10^{-138}$
$2+60\mathrm{i}$	10^{2}	96	$4.5686\ldots\cdot 10^{-225}$
$70\mathrm{i}$	10^{3}	120	$3.9966\ldots\cdot 10^{-1623}$

Table 2.3: Instances of accuracy of (2.10) for discrete circles.

For the sets from Fig. 2.1 and Tables 2.2–2.3 we have

$$Q(\mathfrak{A}) \approx 1. \tag{3.10}$$

Of course, this cannot be true for an arbitrary set $\mathfrak{A}$. To see this, let, for example, $\mathfrak{A}_N(a)$ be the set $\{a, 2a, \ldots, (N-1)a\}$. Then $Q(\mathfrak{A}_N(a))$ is a meromorphic function, different from constant (otherwise we would have a differential equation for $\eta(s)$). Hence, by Picard's theorems, all (except, possibly, one) complex numbers are values of $Q(\mathfrak{A}_N(a))$.

It is not clear in what terms could we describe the class of sets $\mathfrak{A}$ for which

$$|Q(\mathfrak{A}) - 1| < \varepsilon \tag{3.11}$$

for a given positive ε. A more achievable goal could be to find bounds for cases of sets of particular structure, for example, to establish inequalities of the form

$$|Q(\mathfrak{A}_{\mathrm{G}}(a,\delta_1,\delta_2,N_1,N_2)) - 1| < B_{\mathrm{G}}(a,\delta_1,\delta_2,N_1,N_2) \tag{3.12}$$

and

$$|Q(\mathfrak{A}_{\mathrm{C}}(c,r,N) - 1| < B_{\mathrm{C}}(c,r,N) \tag{3.13}$$

with certain functions $B_{\mathrm{G}}(a,\delta_1,\delta_2,N_1,N_2)$ and $B_{\mathrm{C}}(c,r,N)$.

4 Two generalizations

Approximate inequality (3.10) can be generalized in several directions.

4.1 Characteristic polynomials

The determinant of a matrix is just the constant term of its characteristic polynomial; in our notation

$$D_N = C_N(\lambda \to 0). \tag{4.1}$$

n	$V_n(\mathfrak{A}_1)$	$\lvert Q_n(\mathfrak{A}_1)-1\rvert$
0	$-2.8932\ldots\cdot 10^{-48} -5.1873\ldots\cdot 10^{-49}\mathrm{i}$	$1.1085\ldots\cdot 10^{-14}$
1	$1.1853\ldots\cdot 10^{-33} +2.1097\ldots\cdot 10^{-34}\mathrm{i}$	$8.6263\ldots\cdot 10^{-13}$
2	$-2.2690\ldots\cdot 10^{-22} -1.6178\ldots\cdot 10^{-22}\mathrm{i}$	$5.1025\ldots\cdot 10^{-11}$
3	$4.0262\ldots\cdot 10^{-14} +8.4755\ldots\cdot 10^{-14}\mathrm{i}$	$3.6813\ldots\cdot 10^{-9}$
4	$-3.7956\ldots\cdot 10^{-8} +2.0698\ldots\cdot 10^{-7}\mathrm{i}$	$1.7175\ldots\cdot 10^{-7}$
5	$1.5216\ldots\cdot 10^{-3} +8.8240\ldots\cdot 10^{-3}\mathrm{i}$	$8.2908\ldots\cdot 10^{-6}$
6	$3.7984\ldots\cdot 10^{0} -4.2877\ldots\cdot 10^{0}\mathrm{i}$	$1.0541\ldots\cdot 10^{-4}$
7	$-3.0090\ldots\cdot 10^{2} +4.8652\ldots\cdot 10^{2}\mathrm{i}$	$2.3986\ldots\cdot 10^{-3}$
8	$-1.8858\ldots\cdot 10^{3} +4.2625\ldots\cdot 10^{2}\mathrm{i}$	$1.8282\ldots\cdot 10^{-2}$
9	$-1.1127\ldots\cdot 10^{3} +9.0420\ldots\cdot 10^{2}\mathrm{i}$	$5.7323\ldots\cdot 10^{-2}$
10	$-5.2122\ldots\cdot 10^{1} +2.1236\ldots\cdot 10^{1}\mathrm{i}$	$1.0387\ldots\cdot 10^{0}$
11	$1.0390\ldots\cdot 10^{1} -7.6856\ldots\cdot 10^{0}\mathrm{i}$	$5.2691\ldots\cdot 10^{-1}$

Table 4.1: Generalization of (3.10) for grid $\mathfrak{A}_1$ from Fig. 2.1.

In particular, the numerator and the denominator in (3.9) are the constant terms of the following polynomials:

$$C_N\big(x_{j,k} \to \eta^{\langle k\rangle}(a_j)\big) = V_N(\mathfrak{A})\lambda^N + \cdots + V_n(\mathfrak{A})\lambda^n + \cdots + V_0(\mathfrak{A}), \tag{4.2}$$

$$C_N\big(x_{j,0} \to 1,\ x_{j,k} \to \eta^{\langle k\rangle}(a_j)\big) = W_N(\mathfrak{A})\lambda^N + \cdots + W_n(\mathfrak{A})\lambda^n + \cdots + W_0(\mathfrak{A}). \tag{4.3}$$

Let us consider ratios

$$Q_n(\mathfrak{A}) = \frac{V_n(\mathfrak{A})}{W_n(\mathfrak{A})}, \quad n = 0, \ldots N. \tag{4.4}$$

Clearly, $Q_0(\mathfrak{A}) = Q(\mathfrak{A})$, and we have seen that this ratio is close to 1 for the sets from Tables 2.2–2.3 and for sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ from Fig. 2.1. Table 4.1 demonstrates that other ratios $Q_n(\mathfrak{A}_1)$ are also not far from 1 when n is small.

4.2 Missing derivatives

In our search for (an impossible) relation (2.3) we could in advance exclude certain derivatives. More formally, let $\mathfrak{D} = d_0, d_1, \ldots$ be a strictly increasing sequences of non-negative integers and

$$d_0 = 0. \tag{4.5}$$

We consider now the following generalization of (2.4):

$$\eta(a_j) + c_{d_1}(\mathfrak{A},\mathfrak{D})\eta^{\langle d_1\rangle}(a_j) + \cdots + c_{d_{N-1}}(\mathfrak{A},\mathfrak{D})\eta^{\langle d_{N-1}\rangle}(a_j) = b(\mathfrak{A},\mathfrak{D}). \tag{4.6}$$

In the solution of this system (with $j = 0, \ldots, N-1$)

$$b(\mathfrak{A},\mathfrak{D}) = Q(\mathfrak{A},\mathfrak{D}) = \frac{D_N\big(x_{j,k} \to \eta^{\langle d_k\rangle}(a_j)\big)}{D_N\big(x_{j,0} \to 1, x_{j,k} \to \eta^{\langle d_k\rangle}(a_j)\big)}. \tag{4.7}$$

$\mathfrak{M}_{\mathfrak{D}}$	$\lvert Q(\mathfrak{A},\mathfrak{D})-1\rvert$
$\{\}$	$1.4487\ldots\cdot 10^{-26}$
$\{1\}$	$4.5365\ldots\cdot 10^{-25}$
$\{2\}$	$2.1994\ldots\cdot 10^{-25}$
$\{5\}$	$8.0930\ldots\cdot 10^{-26}$
$\{10\}$	$3.5842\ldots\cdot 10^{-26}$
$\{15\}$	$2.1416\ldots\cdot 10^{-26}$
$\{1,2\}$	$7.5493\ldots\cdot 10^{-24}$
$\{2,3\}$	$2.3575\ldots\cdot 10^{-24}$
$\{2,5\}$	$1.3344\ldots\cdot 10^{-24}$
$\{3,7\}$	$5.8436\ldots\cdot 10^{-25}$
$\{2,4,6\}$	$9.1420\ldots\cdot 10^{-24}$
$\{5,11,14\}$	$3.5669\ldots\cdot 10^{-25}$
$\{1,3,5,7\}$	$1.5529\ldots\cdot 10^{-22}$
$\{2,4,6,8\}$	$3.7291\ldots\cdot 10^{-23}$
$\{3,4,5,6,7\}$	$2.1682\ldots\cdot 10^{-22}$
$\{3,6,9,12,15\}$	$1.0883\ldots\cdot 10^{-23}$
$\{9,10,11,12,13\}$	$1.5617\ldots\cdot 10^{-24}$
$\{2,4,6,8,10,12\}$	$3.4899\ldots\cdot 10^{-22}$
$\{1,3,5,7,9,11\}$	$1.8820\ldots\cdot 10^{-21}$
$\{2,4,6,8,10,12,14\}$	$8.5487\ldots\cdot 10^{-22}$
$\{2,3,4,6,8,10,12\}$	$5.6029\ldots\cdot 10^{-21}$

Table 4.2: The distance between $Q(\mathfrak{A},\mathfrak{D})$ and 1 for $\mathfrak{A}=\mathfrak{A}_{\mathrm{C}}(0.6+14\mathrm{i},10,20)$.

A sequence $\mathfrak{D}$ can be defined by the set $\mathfrak{M}_{\mathfrak{D}}$ of non-negative integers missing in $\mathfrak{D}$. Table 4.2 shows how dropping more and more derivatives increases the distance between $Q(\mathfrak{A},\mathfrak{D})$ and 1 for a particular set $\mathfrak{A}$ and a selection of sets $\mathfrak{M}_{\mathfrak{D}}$ of missing derivatives.

5 Dropping $\eta(a)$

Searching for a polynomial

$$P(y_0,y_1,\ldots,y_{N-1}) = c_0y_0+c_1y_1+\cdots+c_{N-1}y_{N-1}-b \tag{5.1}$$

such that

$$P\big(\eta(a),\eta'(a),\ldots,\eta^{\langle N-1\rangle}(a)\big)=0 \tag{5.2}$$

we should exclude the degenerate case of the polynomial being identically equal to 0. In (2.1) this was achieved by fixing $c_0=1$ but now we remove this constraint. Let l be the least index such that $c_l\neq 0$; without loss of generality we put $c_l=1$.

More formally, for a given set $\mathfrak{A} = \{a_0, \dots, a_{N-1}\}$ and non-negative integer l we generalize (2.4) to linear system

$$\eta^{\langle l \rangle}(a_j) + c_{l+1}(\mathfrak{A})\eta^{\langle l+1 \rangle}(a_j) + \ldots + c_{l+N-1}(\mathfrak{A})\eta^{\langle l+N-1 \rangle}(a_j) = b(\mathfrak{A}) \tag{5.3}$$

with $j = 0, \dots, N-1$. We find that the case $l \geq 1$ radically differs from the earlier considered case $l = 0$. Namely, $b(\mathfrak{A})$ has the tendency to be close to 0. In our notation this can be written as

$$D_N\big(x_{j,k} \to \eta^{\langle l+k \rangle}(a_j)\big) \approx 0. \tag{5.4}$$

However, this approximate equality is uninformative. Namely, polynomial D_N has no constant term, so the value of the left-hand side of (5.4) can be small just due to the small values of the eta derivatives.

Nevertheless, approximate equality (5.4) can provide non-trivial information about the relationship between the derivatives of the eta function. To demonstrate this, we select two integers m and n such that $0 \leq m < N$ and $0 \leq n < N$ and define y as the solution of linear equation

$$D_N\big(x_{m,n} \to y,\ x_{j,k} \to \eta^{\langle l+k \rangle}(a_j)\big) = 0. \tag{5.5}$$

In our notation,

$$y = R(l,m,n,\mathfrak{A}) = \frac{D_N\big(x_{m,n} \to 0,\ x_{j,k} \to \eta^{\langle l+k \rangle}(a_j)\big)}{D_N\big(x_{m,n} \to -1,\ x_{m,k} \to 0,\ x_{j,n} \to 0,\ x_{j,k} \to \eta^{\langle l+k \rangle}(a_j)\big)}. \tag{5.6}$$

According to (5.4)–(5.5), we can expect that $y \approx \eta^{\langle l+n \rangle}(a_m)$.

Numerical example 5.1. Let $\mathfrak{A} = \{a_0, \dots, a_{11}\}$ be either of the two sets from Fig. 2.1. We have: for $l = 1, \dots, 11$, $m = 0, \dots, 11$, $n = 0, \dots, 11$

$$\left| \frac{R(l,m,n,\mathfrak{A})}{\eta^{\langle l+n \rangle}(a_m)} - 1 \right| < 10^{-7}. \tag{5.7}$$

Remark 5.1. As in Subsection 4.1 we can consider the characteristic polynomials

$$C_N\big(x_{m,n} \to 0,\ x_{j,k} \to \eta^{\langle l+k \rangle}(a_j)\big) \tag{5.8}$$

and

$$C_N\big(x_{m,n} \to -1,\ x_{m,k} \to 0,\ x_{j,n} \to 0,\ x_{j,k} \to \eta^{\langle l+k \rangle}(a_j)\big) \tag{5.9}$$

and approximate $\eta^{\langle l+n \rangle}(a)$ by ratios of lower coefficients of these polynomials.

Remark 5.2. As in Subsection 4.2 we could drop not only several initial but also some higher derivatives.

Remark 5.3. Similar to (5.8)–(5.9) we can transform (3.9)–(3.10) into linear equation

$$D_N\big(x_{m,n} \to y, x_{j,k} \to \eta^{\langle k \rangle}(a_j)\big) = D_N\big(x_{m,n} \to y, x_{j,0} \to 1, x_{j,k} \to \eta^{\langle k \rangle}(a_j)\big) \tag{5.10}$$

and expect that its solution is close to $\eta^{\langle n \rangle}(a_m)$.

a	$\lvert S_N(a)-1\rvert$		
	$N=20$	$N=70$	$N=200$
-6	$3.9870\ldots\cdot 10^{-18}$	$1.7014\ldots\cdot 10^{-105}$	$3.7560\ldots\cdot 10^{-368}$
-3	$4.7492\ldots\cdot 10^{-22}$	$6.5289\ldots\cdot 10^{-111}$	$6.8468\ldots\cdot 10^{-375}$
-1	$1.2054\ldots\cdot 10^{-24}$	$1.6130\ldots\cdot 10^{-114}$	$2.2057\ldots\cdot 10^{-379}$
$-1+300\mathrm{i}$	$2.8175\ldots\cdot 10^{-16}$	$1.5955\ldots\cdot 10^{-98}$	$1.0048\ldots\cdot 10^{-378}$
-0.8	$6.6429\ldots\cdot 10^{-25}$	$7.0318\ldots\cdot 10^{-115}$	$7.8416\ldots\cdot 10^{-380}$
$-0.8+50\mathrm{i}$	$4.9124\ldots\cdot 10^{-24}$	$1.6522\ldots\cdot 10^{-115}$	$4.8600\ldots\cdot 10^{-380}$
0	$6.1439\ldots\cdot 10^{-26}$	$2.5409\ldots\cdot 10^{-116}$	$1.2527\ldots\cdot 10^{-381}$
$50\mathrm{i}$	$5.1433\ldots\cdot 10^{-25}$	$6.3145\ldots\cdot 10^{-117}$	$7.8169\ldots\cdot 10^{-382}$
$300\mathrm{i}$	$1.4208\ldots\cdot 10^{-17}$	$2.4131\ldots\cdot 10^{-100}$	$6.6292\ldots\cdot 10^{-381}$
$1500\mathrm{i}$	$3.5498\ldots\cdot 10^{-14}$	$3.8431\ldots\cdot 10^{-80}$	$3.1958\ldots\cdot 10^{-307}$
$0.2+10\mathrm{i}$	$2.4939\ldots\cdot 10^{-26}$	$1.0349\ldots\cdot 10^{-116}$	$4.3685\ldots\cdot 10^{-382}$
$0.2+100\mathrm{i}$	$1.6846\ldots\cdot 10^{-21}$	$3.3113\ldots\cdot 10^{-117}$	$8.0859\ldots\cdot 10^{-383}$
0.5	$1.3906\ldots\cdot 10^{-26}$	$3.1908\ldots\cdot 10^{-117}$	$9.4420\ldots\cdot 10^{-383}$
$0.5+50\mathrm{i}$	$1.2520\ldots\cdot 10^{-25}$	$8.2073\ldots\cdot 10^{-118}$	$5.9166\ldots\cdot 10^{-383}$
$0.5+1500\mathrm{i}$	$6.7418\ldots\cdot 10^{-15}$	$1.0285\ldots\cdot 10^{-80}$	$4.0432\ldots\cdot 10^{-308}$
0.8	$5.7069\ldots\cdot 10^{-27}$	$9.1900\ldots\cdot 10^{-118}$	$2.0017\ldots\cdot 10^{-383}$
$0.8+300\mathrm{i}$	$1.1103\ldots\cdot 10^{-18}$	$7.2488\ldots\cdot 10^{-102}$	$1.1928\ldots\cdot 10^{-382}$
$0.8+1500\mathrm{i}$	$2.7648\ldots\cdot 10^{-15}$	$4.2959\ldots\cdot 10^{-81}$	$1.0755\ldots\cdot 10^{-308}$
$1+10\mathrm{i}$	$2.3862\ldots\cdot 10^{-27}$	$3.7524\ldots\cdot 10^{-118}$	$6.9829\ldots\cdot 10^{-384}$
$1+50\mathrm{i}$	$3.0417\ldots\cdot 10^{-26}$	$1.0666\ldots\cdot 10^{-118}$	$4.4787\ldots\cdot 10^{-384}$
$1+300\mathrm{i}$	$5.7549\ldots\cdot 10^{-19}$	$2.9804\ldots\cdot 10^{-102}$	$4.3682\ldots\cdot 10^{-383}$
$2+10\mathrm{i}$	$1.2736\ldots\cdot 10^{-28}$	$5.9438\ldots\cdot 10^{-120}$	$3.9701\ldots\cdot 10^{-386}$
$2+1500\mathrm{i}$	$1.3975\ldots\cdot 10^{-16}$	$7.2077\ldots\cdot 10^{-83}$	$1.7403\ldots\cdot 10^{-311}$

Table 6.1: Numerical data supporting Conjecture 6.1.

Part II
Conjectures

6 First limiting case

Approximate equality (3.10) can be fulfilled with high accuracy when points $a_0,\ldots,a_{N-1}$ are spaced apart (see the cases of large δ_1 or δ_2 in Table 2.2 and [Mat24a, Table 36], and also the cases of large r in Table 2.3 and [Mat24a, Table 37]). But the accuracy of (3.10) can also be high when the points $a_0,\ldots,a_{N-1}$ are close to each other (see the cases when r or both δ_1 and δ_2 are small in the same tables). Thus we can consider the limit of $Q(\mathfrak{A})$ when some of the points from set $\mathfrak{A}$ approach the same limiting values.

In this paper we restrict ourselves to the extreme case when all the points tend to the same number. Let $a_j = a + j\varepsilon$ where ε tends to 0. We have:

$$D_N\big(x_{j,k}\to\eta^{\langle k\rangle}(a_j)\big) = D_N\big(x_{j,k}\to\eta^{\langle j+k\rangle}(a)\big)\varepsilon^{\frac{N(N-1)}{2}} + O(\varepsilon^{\frac{N(N-1)}{2}+1}), \tag{6.1}$$

$$D_N\big(x_{j,0}\to 1, x_{j,k}\to\eta^{\langle k\rangle}(a_j)\big) = D_{N-1}\big(x_{j,k}\to\eta^{\langle j+k+2\rangle}(a)\big)\varepsilon^{\frac{N(N-1)}{2}} + O(\varepsilon^{\frac{N(N-1)}{2}+1}). \tag{6.2}$$

Let

$$S_N(a) = \frac{D_N\big(x_{j,k}\to\eta^{\langle j+k\rangle}(a)\big)}{D_{N-1}\big(x_{j,k}\to\eta^{\langle j+k+2\rangle}(a)\big)}. \tag{6.3}$$

m	$\max_{1\le n\le m+1}\left\|\frac{n-m-2}{n\eta^{\langle l+m\rangle}(a)}\cdot\frac{E_{l,m,N,n-1}(a)}{E_{l,m,N,n}(a)}-1\right\|$ $l=2$	$l=5$	$l=8$
1	$8.336\ldots\cdot 10^{-75}$	$9.685\ldots\cdot 10^{-74}$	$1.023\ldots\cdot 10^{-72}$
2	$6.211\ldots\cdot 10^{-71}$	$6.871\ldots\cdot 10^{-70}$	$6.949\ldots\cdot 10^{-69}$
3	$3.266\ldots\cdot 10^{-67}$	$3.455\ldots\cdot 10^{-66}$	$3.342\ldots\cdot 10^{-65}$
4	$1.296\ldots\cdot 10^{-63}$	$1.312\ldots\cdot 10^{-62}$	$1.213\ldots\cdot 10^{-61}$
5	$4.044\ldots\cdot 10^{-60}$	$3.914\ldots\cdot 10^{-59}$	$3.461\ldots\cdot 10^{-58}$
6	$1.019\ldots\cdot 10^{-56}$	$9.426\ldots\cdot 10^{-56}$	$7.968\ldots\cdot 10^{-55}$
7	$2.114\ldots\cdot 10^{-53}$	$1.868\ldots\cdot 10^{-52}$	$1.510\ldots\cdot 10^{-51}$
8	$3.664\ldots\cdot 10^{-50}$	$3.093\ldots\cdot 10^{-49}$	$2.391\ldots\cdot 10^{-48}$

Table 7.1: Numerical data supporting Conjecture 7.2; $N = 50$, $a = 0.4+17\mathrm{i}$.

From (6.1) and (6.2)

$$S_N(a) = Q\big(\{a_0,\ldots,a_{N-1}\}\big)+O(\varepsilon), \tag{6.4}$$

and according to (3.10) we can expect that $S_N(a)$ is close to 1.

Conjecture 6.1. *Except for countably many values of a,*

$$S_N(a) \xrightarrow[N\to\infty]{} 1, \tag{6.5}$$

where S is defined by (6.3).

Numerical data in Table 6.1 support this conjecture (for other relevant data see [Mat24a, Table 1]).

Remark 6.1. Similar to (5.5), we can consider linear equation in unknown y

$$D_N\big(x_{0,0}\to y,\ x_{j,k}\to\eta^{\langle j+k\rangle}(a)\big) = D_{N-1}\big(x_{j,k}\to\eta^{\langle j+k+2\rangle}(a)\big). \tag{6.6}$$

Conjecture 6.1 suggests that the solution of this equation should be close to $\eta(a)$. Other ways to calculate $\eta(a)$ via derivatives of $\eta(s)$ at $s=a$ were proposed in [Mat23a, Section 6] (see also [Mat23b, Section 8)]).

7 Second limiting case

Now we consider limiting behavior of (5.4). Similar to (6.1) we have:

$$D_N\big(x_{j,k}\to\eta^{\langle l+k\rangle}(a+j\varepsilon)\big) = D_N\big(x_{j,k}\to\eta^{\langle l+j+k\rangle}(a)\big)\varepsilon^{\frac{N(N-1)}{2}}+O(\varepsilon^{\frac{N(N-1)}{2}+1}). \tag{7.1}$$

We can make a guess that

$$D_N\big(x_{j,k}\to\eta^{\langle l+j+k\rangle}(a)\big)\approx 0. \tag{7.2}$$

Like (5.4), by itself this is an uninformative approximate relation among the derivatives of the eta function. To show its non-triviality we can consider counterparts of (5.5), namely, equations

$$D_N\big(j+k=m\Rightarrow x_{j,k}\to y, x_{j,k}\to\eta^{\langle l+j+k\rangle}(a)\big)=0 \tag{7.3}$$

for $m = 0,\dots,2N-2$. However, there is an essential distinction: (5.4) is a linear equation in y but (7.3) is linear in two extreme cases only, namely, when $m = 0$ and when $m = 2N-2$. For these values of m the solution is given by

$$y = \frac{D_N\big(x_{m',m'} \to 0,\ x_{j,k} \to \eta^{\langle l+j+k\rangle}(a)\big)}{D_N\big(x_{m',m'} \to -1,\ x_{m',k} \to 0,\ x_{j,m'} \to 0,\ x_{j,k} \to \eta^{\langle l+j+k\rangle}(a)\big)} \tag{7.4}$$

where $m' = m/2$.

In general case, the degree of equation (7.3) is equal to

$$d = \min(m+1, 2N-m-1). \tag{7.5}$$

Let

$$\begin{aligned} E_{l,m,N}(a,y) &= D_N\big(j+k=m \Rightarrow x_{j,k} \to y,\ x_{j,k} \to \eta^{\langle l+j+k\rangle}(a)\big) \\ &= E_{l,m,N,d}(a)y^d + \cdots + E_{l,m,N,n}(a)y^n + \cdots + E_{l,m,N,0}(a). \end{aligned} \tag{7.6}$$

Assuming 7.2, we can expect that *at least one* of the roots of equation (7.3) is close to $\eta^{\langle l+m\rangle}(a)$. Calculations show that when d is relatively small to N, *all* roots of equation (7.3) are close to $\eta^{\langle l+m\rangle}(a)$.

Here we consider only the case when d small due to the small value of m (for the case when m is close to $2N-2$ see [Mat24a, Subsection 3.4]).

Conjecture 7.1. *Except for countably many values of a, for every positive integer l and non-negative integer m the maximal distance between the roots of the polynomial $E_{l,m,N}(a,y)$ (defined by (7.6)) and $\eta^{\langle l+m\rangle}(a)$ tends to zero as $N \to \infty$.*

This conjecture does not have the “canonical” form (1.5), and we do not present here numerical examples which would support it (for such data see [Mat24a, Tables 11-16]. Instead of this, we deduce several corollaries of the form (1.5). Taken together, they are essentially equivalent to Conjecture 7.1.

Namely, Conjecture 7.1 implies that if m is small (relative to N) then polynomial $E_{l,m,N}(a,y)$ should have a special structure, namely,

$$E_{l,m,N}(a,y) \approx A(y - \eta^{\langle l+m\rangle}(a))^d \tag{7.7}$$

for some non-zero constant A. This opens several ways to calculate (approximation to) $\eta^{\langle m\rangle}(a)$ via the values of other derivatives.

Conjecture 7.2. *Except for countably many values of a, for every positive integer l, non-negative integer m and an n such that $0 < n \le m+1$*

$$\frac{n-m-2}{n} \cdot \frac{E_{l,m,N,n-1}(a)}{E_{l,m,N,n}(a)} \xrightarrow[N\to\infty]{} \eta^{\langle l+m\rangle}(a), \tag{7.8}$$

where $E_{l,m,N,n}$ is defined by (7.6).

For numerical data supporting Conjecture 7.2 see Table 7.1 and [Mat24a, Tables 17–19].

Remark 7.1. Assuming Conjecture 6.1, we can consider equations

$$\begin{aligned} &D_N\big(j+k=m \Rightarrow x_{j,k} \to y,\ x_{j,k} \to \eta^{\langle j+k\rangle}(a)\big) \\ &\quad = D_{N-1}\big(j+k+2=m \Rightarrow x_{j,k} \to y,\ x_{j,k} \to \eta^{\langle j+k+2\rangle}(a)\big) \end{aligned} \tag{7.9}$$

and state for them analogs of Conjectures 7.1 and 7.2 (for numerical data see [Mat24a, Tables 7-10].

a	$\lvert S_{!N}(a)-1\rvert$		
	$N=70$	$N=200$	$N=700$
-2	$6.6474\ldots\cdot 10^{-3}$	$4.2041\ldots\cdot 10^{-5}$	$1.0842\ldots\cdot 10^{-13}$
-1	$9.9362\ldots\cdot 10^{-3}$	$8.6712\ldots\cdot 10^{-6}$	$1.8735\ldots\cdot 10^{-14}$
-0.8	$1.1821\ldots\cdot 10^{-2}$	$1.0637\ldots\cdot 10^{-5}$	$1.2612\ldots\cdot 10^{-14}$
0	$1.2694\ldots\cdot 10^{-2}$	$1.5834\ldots\cdot 10^{-6}$	$2.3561\ldots\cdot 10^{-15}$
$50\mathrm{i}$	$9.9046\ldots\cdot 10^{-2}$	$5.6900\ldots\cdot 10^{-5}$	$1.0808\ldots\cdot 10^{-14}$
0.2	$3.9934\ldots\cdot 10^{-3}$	$8.0428\ldots\cdot 10^{-7}$	$3.7177\ldots\cdot 10^{-14}$
$0.2+10\mathrm{i}$	$9.6420\ldots\cdot 10^{-4}$	$1.1453\ldots\cdot 10^{-6}$	$4.8160\ldots\cdot 10^{-15}$
0.5	$1.2866\ldots\cdot 10^{-3}$	$3.1413\ldots\cdot 10^{-7}$	$8.2725\ldots\cdot 10^{-16}$
0.8	$4.8940\ldots\cdot 10^{-4}$	$1.6870\ldots\cdot 10^{-7}$	$3.1938\ldots\cdot 10^{-16}$
$0.8+10\mathrm{i}$	$2.5509\ldots\cdot 10^{-4}$	$2.6686\ldots\cdot 10^{-7}$	$7.1105\ldots\cdot 10^{-16}$
$1+10\mathrm{i}$	$3.4554\ldots\cdot 10^{-4}$	$2.3766\ldots\cdot 10^{-7}$	$4.4841\ldots\cdot 10^{-16}$
$1+50\mathrm{i}$	$8.7300\ldots\cdot 10^{-2}$	$4.6678\ldots\cdot 10^{-6}$	$4.5817\ldots\cdot 10^{-16}$
$2+10\mathrm{i}$	$2.4417\ldots\cdot 10^{-4}$	$1.4395\ldots\cdot 10^{-7}$	$2.5638\ldots\cdot 10^{-17}$

Table 8.1: Numerical data supporting Conjecture 8.1.

a	$\lvert S^{!}_{N}(a)-1\rvert$		
	$N=70$	$N=200$	$N=700$
-2	$2.2873\ldots\cdot 10^{-1}$	$2.9563\ldots\cdot 10^{-2}$	$1.0597\ldots\cdot 10^{-2}$
-1	$4.0733\ldots\cdot 10^{-2}$	$1.8496\ldots\cdot 10^{-2}$	$6.7039\ldots\cdot 10^{-3}$
-0.8	$3.6069\ldots\cdot 10^{-2}$	$2.0192\ldots\cdot 10^{-2}$	$5.9950\ldots\cdot 10^{-3}$
0	$2.3313\ldots\cdot 10^{-2}$	$1.1531\ldots\cdot 10^{-2}$	$4.1362\ldots\cdot 10^{-3}$
$50\mathrm{i}$	$7.2888\ldots\cdot 10^{-2}$	$2.7648\ldots\cdot 10^{-2}$	$9.1542\ldots\cdot 10^{-3}$
0.2	$2.0458\ldots\cdot 10^{-2}$	$6.6277\ldots\cdot 10^{-3}$	$3.7877\ldots\cdot 10^{-3}$
$0.2+10\mathrm{i}$	$6.8470\ldots\cdot 10^{-2}$	$3.0200\ldots\cdot 10^{-2}$	$1.1262\ldots\cdot 10^{-2}$
0.5	$1.5076\ldots\cdot 10^{-2}$	$7.6916\ldots\cdot 10^{-3}$	$3.0248\ldots\cdot 10^{-3}$
0.8	$1.3358\ldots\cdot 10^{-2}$	$6.7687\ldots\cdot 10^{-3}$	$6.9181\ldots\cdot 10^{-4}$
$0.8+10\mathrm{i}$	$3.8167\ldots\cdot 10^{-2}$	$1.7013\ldots\cdot 10^{-2}$	$6.3566\ldots\cdot 10^{-3}$
$1+50\mathrm{i}$	$2.1107\ldots\cdot 10^{-2}$	$8.5441\ldots\cdot 10^{-3}$	$2.9618\ldots\cdot 10^{-3}$
$2+10\mathrm{i}$	$1.2910\ldots\cdot 10^{-2}$	$5.7831\ldots\cdot 10^{-3}$	$2.1859\ldots\cdot 10^{-3}$

Table 8.2: Numerical data supporting Conjecture 8.2.

8 Modifications

Besides (6.5) and (7.8) (and corresponding generalizations indicated in the remarks), one can state similar conjectures of the form (1.5) for a great number of other ratios of polynomial. Here we indicate a few directions of possible modifications.

8.1 Taylor coefficients

Let

$$D_{!N} = D_N(x_{j,k} \to x_{j,k}/k!). \tag{8.1}$$

By analogy with (6.3) we define

$$S_{!N}(a) = \frac{D_{!N}\big(x_{j,k} \to \eta^{\langle j+k\rangle}(a)\big)}{D_{!N-1}\big(x_{j,k} \to \eta^{\langle j+k+2\rangle}(a)\big)} \tag{8.2}$$

and state a counterpart of Conjecture 6.1.

Conjecture 8.1. *Except for countably many values of a,*

$$S_{!N}(a) \xrightarrow[N\to\infty]{} 1. \tag{8.3}$$

Comparing values of $S_{!N}(a)$ in Table 8.1 (see also [Mat24a, Tables 27]) with values of $S_N(a)$ in Table 6.1 for the same N and a, we should observe that convergence in (8.3) is much slower than in (6.5).

8.2 "Anti-Taylor" coefficients

Similar to (8.1) and (8.2), let

$$D^{!}_{N} = D_N(x_{j,k} \to k!x_{j,k}) \tag{8.4}$$

and

$$S^{!}_{N}(a) = \frac{D^{!}_{N}\big(x_{j,k} \to \eta^{\langle j+k\rangle}(a)\big)}{D^{!}_{N-1}\big(x_{j,k} \to \eta^{\langle j+k+2\rangle}(a)\big)}. \tag{8.5}$$

Conjecture 8.2. *Except for countably many values of a,*

$$S^{!}_{N}(a) \xrightarrow[N\to\infty]{} 1. \tag{8.6}$$

Comparing values of $S^{!}_{N}(a)$ in Table 8.2 (see also [Mat24a, Tables 29]) with values of $S_{!N}(a)$ in Table 8.1 for the same N and a, we should observe that convergence in (8.6) is even slower than in (8.3).

8.3 Minors

In this subsection we extend Conjecture 6.1. Let

$$\mathfrak{D} = d_0, d_1, \dots \tag{8.7}$$

and

$$\mathfrak{F} = f_0, f_1 \dots \tag{8.8}$$

be two strictly increasing sequences of non-negative integers. We generalize (3.1) to

$$D_{N,\mathfrak{D},\mathfrak{F}} = \begin{vmatrix} x_{f_0,d_0} & \dots & x_{f_0,d_{N-1}} \\ \vdots & \ddots & \vdots \\ x_{f_{N-1},d_0} & \dots & x_{f_{N-1},d_{N-1}} \end{vmatrix}. \tag{8.9}$$

As in Subsection 4.2, $\mathfrak{M}_{\mathfrak{D}}$ and $\mathfrak{M}_{\mathfrak{F}}$ denote the sets of non-negative integers missing in $\mathfrak{D}$ and $\mathfrak{F}$ respectively.

Conjecture 8.3. *Let $\mathfrak{D}_1 = d_1, d_2, \dots$ and $\mathfrak{F}_1 = f_1, f_2 \dots$ be the tailors of sequences (8.7) and (8.8). If $d_0 = f_0 = 0$ and sets $\mathfrak{M}_{\mathfrak{D}}$ and $\mathfrak{M}_{\mathfrak{F}}$ are finite, then except for countably many values of a,*

$$S_{N,\mathfrak{D},\mathfrak{F}}(a) = \frac{D_{N,\mathfrak{D},\mathfrak{F}}\big(x_{j,k} \to \eta^{\langle j+k\rangle}(a)\big)}{D_{N-1,\mathfrak{D}_1,\mathfrak{F}_1}\big(x_{j,k} \to \eta^{\langle j+k+2\rangle}(a)\big)} \xrightarrow[N\to\infty]{} 1. \tag{8.10}$$

$\mathfrak{M}_{\mathfrak{F}}$	$\mathfrak{M}_{\mathfrak{D}}$	$S_{N,\mathfrak{D},\mathfrak{F}}(a)$ $N=40$	$S_{N,\mathfrak{D},\mathfrak{F}}(a)$ $N=150$
$\{1\}$	$\{1\}$	$1.5705\ldots\cdot10^{-57}$	$4.1503\ldots\cdot10^{-273}$
	$\{2,3\}$	$1.5883\ldots\cdot10^{-56}$	$1.4302\ldots\cdot10^{-271}$
	$\{1,2,3\}$	$1.0588\ldots\cdot10^{-54}$	$3.0229\ldots\cdot10^{-269}$
	$\{4,5\}$	$4.4938\ldots\cdot10^{-57}$	$4.2342\ldots\cdot10^{-272}$
	$\{2,3,5\}$	$1.9855\ldots\cdot10^{-55}$	$5.9628\ldots\cdot10^{-270}$
	$\{1,3,4,5,7\}$	$6.3476\ldots\cdot10^{-53}$	$1.9159\ldots\cdot10^{-266}$
$\{2,3\}$	$\{1\}$	$1.5883\ldots\cdot10^{-56}$	$1.4302\ldots\cdot10^{-271}$
	$\{2,3\}$	$1.6076\ldots\cdot10^{-55}$	$4.9296\ldots\cdot10^{-270}$
	$\{1,2,3\}$	$1.0726\ldots\cdot10^{-53}$	$1.0420\ldots\cdot10^{-267}$
	$\{4,5\}$	$4.5482\ldots\cdot10^{-56}$	$1.4593\ldots\cdot10^{-270}$
	$\{2,3,5\}$	$2.0112\ldots\cdot10^{-54}$	$2.0554\ldots\cdot10^{-268}$
	$\{1,3,4,5,7\}$	$6.4409\ldots\cdot10^{-52}$	$6.6060\ldots\cdot10^{-265}$
$\{1,2,3\}$	$\{1\}$	$1.0588\ldots\cdot10^{-54}$	$3.0229\ldots\cdot10^{-269}$
	$\{2,3\}$	$1.0726\ldots\cdot10^{-53}$	$1.0420\ldots\cdot10^{-267}$
	$\{1,2,3\}$	$7.1626\ldots\cdot10^{-52}$	$2.2028\ldots\cdot10^{-265}$
	$\{4,5\}$	$3.0343\ldots\cdot10^{-54}$	$3.0848\ldots\cdot10^{-268}$
	$\{2,3,5\}$	$1.3430\ldots\cdot10^{-52}$	$4.3452\ldots\cdot10^{-266}$
	$\{1,3,4,5,7\}$	$4.3085\ldots\cdot10^{-50}$	$1.3968\ldots\cdot10^{-262}$
$\{4,5\}$	$\{1\}$	$4.4938\ldots\cdot10^{-57}$	$4.2342\ldots\cdot10^{-272}$
	$\{2,3\}$	$4.5482\ldots\cdot10^{-56}$	$1.4593\ldots\cdot10^{-270}$
	$\{1,2,3\}$	$3.0343\ldots\cdot10^{-54}$	$3.0848\ldots\cdot10^{-268}$
	$\{4,5\}$	$1.2867\ldots\cdot10^{-56}$	$4.3204\ldots\cdot10^{-271}$
	$\{2,3,5\}$	$5.6898\ldots\cdot10^{-55}$	$6.0849\ldots\cdot10^{-269}$
	$\{1,3,4,5,7\}$	$1.8219\ldots\cdot10^{-52}$	$1.9556\ldots\cdot10^{-265}$
$\{2,3,5\}$	$\{1\}$	$1.9855\ldots\cdot10^{-55}$	$5.9628\ldots\cdot10^{-270}$
	$\{2,3\}$	$2.0112\ldots\cdot10^{-54}$	$2.0554\ldots\cdot10^{-268}$
	$\{1,2,3\}$	$1.3430\ldots\cdot10^{-52}$	$4.3452\ldots\cdot10^{-266}$
	$\{4,5\}$	$5.6898\ldots\cdot10^{-55}$	$6.0849\ldots\cdot10^{-269}$
	$\{2,3,5\}$	$2.5183\ldots\cdot10^{-53}$	$8.5712\ldots\cdot10^{-267}$
	$\{1,3,4,5,7\}$	$8.0783\ldots\cdot10^{-51}$	$2.7554\ldots\cdot10^{-263}$
$\{1,3,4,5,7\}$	$\{1\}$	$6.3476\ldots\cdot10^{-53}$	$1.9159\ldots\cdot10^{-266}$
	$\{2,3\}$	$6.4409\ldots\cdot10^{-52}$	$6.6060\ldots\cdot10^{-265}$
	$\{1,2,3\}$	$4.3085\ldots\cdot10^{-50}$	$1.3968\ldots\cdot10^{-262}$
	$\{4,5\}$	$1.8219\ldots\cdot10^{-52}$	$1.9556\ldots\cdot10^{-265}$
	$\{2,3,5\}$	$8.0783\ldots\cdot10^{-51}$	$2.7554\ldots\cdot10^{-263}$
	$\{1,3,4,5,7\}$	$2.6003\ldots\cdot10^{-48}$	$8.8624\ldots\cdot10^{-260}$

Table 8.3: Numerical data supporting Conjecture 8.3; $a=0.4+16\mathrm{i}$.

n	$\left\lvert \frac{V_{l,n,\alpha}(a)}{\eta^{\langle l\rangle}(a)W_{l,n,\alpha}(a)} - 1 \right\rvert$ $\alpha = \alpha_{17,3}$	$\alpha = \alpha_{19,3}$	$\alpha = \alpha_{23,3}$
0	$4.981\ldots\cdot 10^{-22}$	$3.585\ldots\cdot 10^{-25}$	$1.236\ldots\cdot 10^{-31}$
1	$2.100\ldots\cdot 10^{-19}$	$1.721\ldots\cdot 10^{-22}$	$7.387\ldots\cdot 10^{-29}$
2	$4.741\ldots\cdot 10^{-17}$	$4.302\ldots\cdot 10^{-20}$	$2.314\ldots\cdot 10^{-26}$
3	$2.157\ldots\cdot 10^{-14}$	$3.209\ldots\cdot 10^{-17}$	$2.695\ldots\cdot 10^{-23}$
4	$6.251\ldots\cdot 10^{-12}$	$2.914\ldots\cdot 10^{-14}$	$6.718\ldots\cdot 10^{-21}$
5	$1.049\ldots\cdot 10^{-9}$	$2.096\ldots\cdot 10^{-12}$	$8.715\ldots\cdot 10^{-19}$
6	$7.331\ldots\cdot 10^{-8}$	$5.602\ldots\cdot 10^{-11}$	$9.066\ldots\cdot 10^{-16}$
7	$6.999\ldots\cdot 10^{-7}$	$1.894\ldots\cdot 10^{-8}$	$2.404\ldots\cdot 10^{-13}$
8	$6.011\ldots\cdot 10^{-4}$	$1.650\ldots\cdot 10^{-6}$	$5.683\ldots\cdot 10^{-11}$

Table 8.4: Approximation of $\eta^{\langle l\rangle}(a)$ by ratios of the initial coefficients of polynomials (8.13) and (8.14); $l = 2$, $a = 0.7 + 19\mathrm{i}$.

Table 8.3 presents data supporting Conjecture 8.3 (see also [Mat24a, Table 32]).

Remark 8.1. Very probably, the requirement of the finiteness of sets $\mathfrak{M}_{\mathfrak{D}}$ and $\mathfrak{M}_{\mathfrak{F}}$ is excessive. Most likely, it can be replaced by certain restrictions on the growth of sequences (8.7) and (8.8) (cf. [Mat24a, Tables 33–34]).

8.4 Generalized characteristic polynomials

Let

$$\alpha = \langle g_0, \ldots, g_{N-1} \rangle \tag{8.11}$$

be a permutation of numbers $0, \ldots, N-1$. We generalize (3.7) to

$$C_\alpha = D_N(j = g_k \Rightarrow x_{j,k} \to x_{j,k} - \lambda). \tag{8.12}$$

Similar to (4.1), D_N is the constant term of C_α up to the sign (depending on the parity of α). Respectively, for $m = 0$ ratio (7.4) is equal to the ratio of the constant terms of polynomials

$$\begin{aligned} &C_\alpha\big(x_{0,0} \to 0,\ x_{j,k} \to \eta^{\langle l+j+k\rangle}(a)\big) \\ &\qquad = V_{l,N,\alpha}(a)\lambda^N + \cdots + V_{l,n,\alpha}(a)\lambda^n + \cdots + V_{l,0,\alpha}(a), \end{aligned} \tag{8.13}$$

$$\begin{aligned} &C_\alpha\big(x_{0,0} \to -1,\ x_{0,k} \to 0,\ x_{j,0} \to 0,\ x_{j,k} \to \eta^{\langle l+j+k\rangle}(a)\big) \\ &\qquad = W_{l,N,\alpha}(a)\lambda^N + \cdots + W_{l,n,\alpha}(a)\lambda^n + \cdots + W_{l,0,\alpha}(a). \end{aligned} \tag{8.14}$$

For relatively prime positive integers q and r let $\alpha_{q,r}$ denote such permutation (8.11) that

$$g_k \equiv kr \pmod{q}. \tag{8.15}$$

Similar to Table 4.1, we observe in Table 8.4 that for small n ratios $V_{l,n,\alpha}(a)/W_{l,n,\alpha}(a)$ are close to $\eta^{\langle l\rangle}(a)$. For other relevant data see [Mat24a, Subsection 4.6].

9 Concluding remark

All our numerical data were for the eta function. Very likely, other functions defined by Dirichlet series have similar properties.

Of course, the most interesting case is the classic Riemann zeta function. Calculations show that for this function we can observe phenomena similar to those discussed in Part I. However, their accuracy can be rather low when some elements of $\mathfrak{A}$ are close to the pole of the zeta function. Calculations performed so far do not allow us to put forward counterparts of Conjectures 6.1–8.3 (for the relevant numerical data see [Mat24a, Tables 39–40]).

Acknowledgments

Numerical data for all the Tables were computed by programs written in JULIA programming language [BEKS17] with package NEMO [FHHJ17].

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