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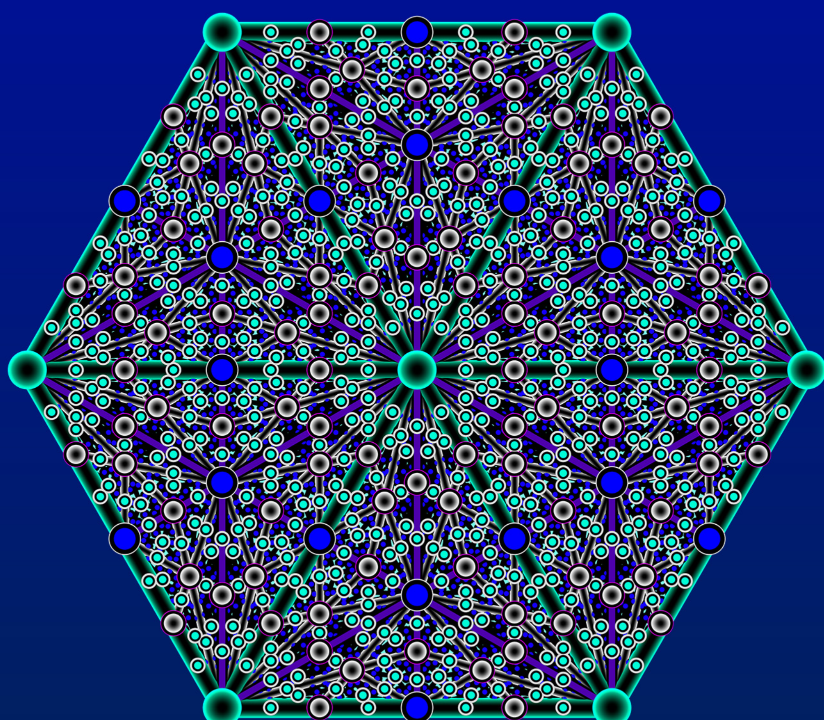

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A hyperelliptic saga on a generating function of the squares of Legendre polynomials

Mark H. F. van Hoeij * Duco van Straten † Wadim Zudilin

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Abstract:
We decompose the generating function $\sum_{n=0}^{\infty}\binom{2n}{n}P_n(y)^2z^n$ of the squares of Legendre polynomials as a product of periods of hyperelliptic curves. These periods satisfy a family of *second* order differential equations. This is highly unusual, since *four* is the expected order for genus 2. These second order equations are arithmetic and yet, surprisingly, their monodromy group is dense in $\mathrm{SL}_2(\mathbb{R})$. This suggests that they cannot be solved in terms of hypergeometric functions, which is novel for arithmetic second order differential equations that are *defined over* $\mathbb{Q}$, and also novel for a *family* of such equations. We complement our analysis with a recipe for constructing similar examples.

Key words and phrases:
Picard–Fuchs differential operator; Integrality; Hyperelliptic curve; Humbert's equation; Legendre polynomial

1 Introduction

It would not be mistaken to call Legendre polynomials

$$P_n(y) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left((y^2-1)^n\right) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{y-1}{2}\right)^k$$

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the most classical *special* polynomials. They can also be given by a recurrence equation

$$\begin{gathered}(n+1)P_{n+1}(y)-(2n+1)yP_n(y)+nP_{n-1}(y)=0 \quad \text{for } n\geq 0,\\ \text{with } P_0(y)=1 \text{ and } P_{-1}(y)=0,\end{gathered} \tag{1.1}$$

or by a generating function

$$F_P(y,z)=\sum_{n=0}^{\infty}P_n(y)z^n=\frac{1}{\sqrt{1-2yz+z^2}}. \tag{1.2}$$

Though many features of Legendre polynomials appear in more general families of special polynomials, there are still surprises kept by and open questions for this classic.

Legendre polynomials can also be expressed with Euler–Gauss hypergeometric functions

$${}_2F_1\left(\begin{matrix}a,\,b\\ c\end{matrix}\,\middle|\,x\right)={}_2F_1(a,b;c\mid x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!\,(c)_n}x^n,$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)=\prod_{j=0}^{n-1}(a+j)$ is Pochhammer's symbol. This is true for the polynomials themselves

$$P_n(y)={}_2F_1\left(\begin{matrix}-n,\,n+1\\ 1\end{matrix}\,\middle|\,\frac{1-y}{2}\right)=\left(\frac{y+1}{2}\right)^n{}_2F_1\left(\begin{matrix}-n,\,-n\\ 1\end{matrix}\,\middle|\,\frac{y-1}{y+1}\right),$$

as well as for the generating function of their squares

$$F_{P^2}(y,z)=\sum_{n=0}^{\infty}P_n(y)^2z^n=\frac{1}{\sqrt{1-2y^2z+z^2}}\cdot{}_2F_1\left(\begin{matrix}\frac14,\,\frac34\\ 1\end{matrix}\,\middle|\,\frac{4(1-y^2)^2z^2}{(1-2y^2z+z^2)^2}\right) \tag{1.3}$$

extracted from [Zud14] (for which another 'more radical' formula was given by W. N. Bailey in the 1940s). Some recent arithmetic questions, which we briefly outline below, suggested the existence of 'explicit' expressions for the twist

$$\tilde{F}_{P^2}(y,z)=\sum_{n=0}^{\infty}\binom{2n}{n}P_n(y)^2z^n \tag{1.4}$$

and for the generating function of the cubes

$$F_{P^3}(y,z)=\sum_{n=0}^{\infty}P_n(y)^3z^n. \tag{1.5}$$

In spite of considerable efforts, no such formulas had been found.

Apart from (1.3), other reasons to expect existence of ${}_2F_1$ type expressions for $\tilde{F}_{P^2}(y,z)$ and $F_{P^3}(y,z)$ are based on similarities the two share with one of Appell's bivariate hypergeometric functions,

$$\mathrm{F}_4\left(\begin{matrix}a,\,b\\ c_1,c_2\end{matrix}\,\middle|\,z,w\right)=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c_1)_m(c_2)_n}z^mw^n. \tag{1.6}$$

For the latter, we have the celebrated Barnes–Bailey identity [Bai33, Bai34, Bai54] (another classic!)

$$\mathrm{F}_4\left(\begin{matrix} a, b \\ c_1, c_2 \end{matrix} \,\middle|\, x(1-y), y(1-x)\right) = {}_2F_1\left(\begin{matrix} a, b \\ c_1 \end{matrix} \,\middle|\, x\right) {}_2F_1\left(\begin{matrix} a, b \\ c_2 \end{matrix} \,\middle|\, y\right) \tag{1.7}$$

when $c_1 + c_2 = a + b + 1$ (see [Zud19, Section 3]). The functions (1.4), (1.5) and (1.6) (for special choices of a, b and $c_1 = c_2 = 1$) show up in an arithmetic context of formulae for π, which were experimentally discovered by Zhi-Wei Sun [Sun20]. Resolutions for related conjectures from [Sun20] make use of the decomposition (1.7) and its numerous generalizations — see [CWZ13, CWZ18, RS13, Wan14, WZ12, WY22]. At the same time this methodology does not seem to extend to examples from [Sun20] that involve $\tilde{F}_{P^2}(y,z)$ and $F_{P^3}(y,z)$. Proofs of some formulae for π with $\tilde{F}_{P^2}(y,z)$ are given in [Zud14]; the crucial ingredient of those is a formula for (1.4) in a special case $z = 4(y^2-1)/(y^2+3)^2$.

For all these reasons, our initial goal was a general formula for $\tilde{F}_{P^2}(y,z)$ with a product like (1.7). And while we did find a product whose factors satisfy second-order equations, they did not turn out to be ${}_2F_1$ expressible. Instead of elliptic integrals (very common for arithmetic second-order equations) for the factors, we were surprised to witness *hyperelliptic* integrals.

Theorem 1.1. *The following formula holds*:

$$\tilde{F}_{P^2}(y,z) = w\, I_+(4z, w^2) I_-(4z, w^2) \tag{1.8}$$

where

$$w = \sqrt{(1+4z)^2 - 16y^2 z} + 4y\sqrt{-z}$$

and

$$I_\pm(u,x) = \frac{1}{\pi}\int_0^1 \frac{1 - uv \pm v\sqrt{2u^2 - 2u}}{\sqrt{v(1-v)\big((1-v)(1-u^2v)(1+uv)^2 + xv(1-uv)^2\big)}}\,\mathrm{d}v. \tag{1.9}$$

Proof. It suffices to compute a differential equation for both sides of (1.8) and then compare initial conditions. (The equation for $I_\pm(u,x)$ is in Section 3.) □

While this proof is technically valid, two issues remain. First, it does not explain *how the formula was found* — and this was not easy! — the details are in Section 2 (which also serves as a more detailed proof). Second, and more important, the proof ignores the fascinating properties of the objects involved. The geometry behind the second-order linear differential equation satisfied by $I_\pm(u,x)$ is quite remarkable. The integral in (1.9) represents periods of hyperelliptic curves, of genus 2, for which it is highly unusual to satisfy *second-order* linear differential equations since *four* is the expected order. These equations are arithmetic[1] and yet, their monodromy group is *dense* in $\mathrm{SL}_2(\mathbb{R})$. This suggests that they cannot be solved in terms of ${}_2F_1$ hypergeometric functions, which is novel for an arithmetic second-order equation that is defined over $\mathbb{Q}$. Indeed, in the OEIS [ST] there are no examples yet[2] of a_n in $\mathbb{Z}$ with $f(x) = \sum_{n=0}^\infty a_n x^n$ holonomic of order 2, a positive radius of convergence, where $f(x)$ is not expressible in terms of algebraic functions, exp, log, ${}_2F_1$, etc. This is despite the fact that the database [ST] records hundreds of non-trivial examples that are arithmetic and differential order 2.

[1] Here, arithmeticity of a differential equation refers to the existence of a globally bounded solution, that is, of a power series solution $f(x) = \sum_{n=0}^\infty a_n x^n \in 1 + x\mathbb{Q}[[x]]$ such that $f(cx) \in \mathbb{Z}[[x]]$ for some non-zero constant c.

[2] At the time of writing, but this can now be changed by uploading an instance of Theorem 2.1 or Remark 5.1!

Being defined over $\mathbb{Q}$ and having a dense monodromy group (Theorem 4.1) are not the only novelties in our arithmetic non-${}_2F_1$ example. Also novel is that it is not an isolated equation, but rather a family of equations, in which the 4 non-apparent singularities can move freely.[3] This discovery of a new class of arithmetic second-order differential equations led to a considerable expansion of what was originally planned to be an ordinary mathematics paper finishing about the place where Section 2 ends. The current exposition includes a discussion of the monodromy of the differential operator encountered and its explicit connection with the geometry of genus 2 curves with a specific feature — real multiplication. This is done in Sections 3 and 4, while Section 5 compares our example to prior counterexamples to a conjecture of Dwork.

For the generating function (1.5) we give an identity of different sort in Section 6. Perhaps a decomposition of (1.5) similar to the one in Theorem 1.1 exists, but over an elliptic extension of $\mathbb{Q}(z)$. Finally, Section 7 summarizes directions for further exploration of the novelty.

Remark 1.1 (analytic aspects of formula (1.8))**.** For fixed $y \in \mathbb{C}$, the series $\tilde{F}_{P^2}(y,z) \in \mathbb{C}[[z]]$ has radius of convergence

$$R_y = \frac{\min(|y \pm \sqrt{y^2-1}|)^2}{4}.$$

We have $R_y > 0$ and $R_y \le 1/4$ for all $y \in \mathbb{C}$, with equality if and only if y is a real number in the interval $[-1,1]$. Theorem 1.1 holds for all $y,z \in \mathbb{R}$ with $|z| < R_y$. It also holds for $y,z \in \mathbb{C}$ if $|z|$ is small enough to ensure that the real parts of $(1+4z)^2 - 16y^2z = 1 + \mathcal{O}(z)$ and $(1-v)(1-u^2v)(1+uv)^2 + xv(1-uv)^2 = 1 + \mathcal{O}(z^{1/2})$ are positive for any $v \in [0,1]$ (here $u = 4z$ and $x = w^2$). This restriction on $|z|$ ensures that two of the square roots in Theorem 1.1 do not suddenly change sign (the signs of $\sqrt{-z}$ and $\sqrt{2u^2-2u}$ do not affect the validity of (1.8)).

2 How the formula was found

Throughout this section, we write $F(y,z)$ for $\tilde{F}_{P^2}(y,z)$. This function satisfies the following differential system of holonomic rank 4

$$z\left(y^2 - 2z - \frac{1}{2}\right)\frac{\partial^2 F}{\partial y\,\partial z} + \frac{y}{2}(1-y^2)\frac{\partial^2 F}{\partial y^2} - (y^2+z)\frac{\partial F}{\partial y} + yz\frac{\partial F}{\partial z} = 0,$$

$$F + \left(2z^2 - \frac{z}{2}\right)\frac{\partial^2 F}{\partial z^2} + \left(5z - \frac{1}{2}\right)\frac{\partial F}{\partial z} + \left(y - \frac{1}{y}\right)\left(\left(z + \frac{1}{4}\right)\frac{\partial^2 F}{\partial y\,\partial z} + \frac{1}{2}\frac{\partial F}{\partial y}\right) = 0$$

obtained from [Zud14, eq. (1.4)],

$$F(y,z) = \sum_{n=0}^{\infty}\binom{2n}{n}z^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\binom{2k}{k}\left(\frac{y^2-1}{4}\right)^k,$$

with creative telescoping. After eliminating derivatives with respect to one variable we obtain fourth-order linear differential equations $\mathrm{DE}_y \in \mathbb{Q}(y,z)[\mathrm{d}/\mathrm{d}y]$ and $\mathrm{DE}_z \in \mathbb{Q}(y,z)[\mathrm{d}/\mathrm{d}z]$. They are too large to print here but are available on our website [vH23]. One can also construct DE_z by using `Maple`'s `gfun` to convert a recurrence for $\binom{2n}{n}P_n(y)^2$ to a differential equation.

[3] This requires $4-3=1$ parameter since one can freely move 3 singularities with a Möbius transformation.

Definition 2.1. For a differential operator L, let $V(L)$ denote the solution space of L in a universal extension. The *least common left multiple* $\mathrm{LCLM}(L_1, L_2)$ is the lowest-order differential operator L with $y_1 + y_2 \in V(L)$ for any $y_1 \in V(L_1)$, $y_2 \in V(L_2)$. The *symmetric product* $L_1 \circledP L_2$ is defined similarly, just replace $y_1 + y_2$ by $y_1 \cdot y_2$.

2.1 Decomposing DE_z

The first step to solving a differential equation is to check if it factors [vH96], however, DE_y and DE_z are irreducible. The next step is the algorithm from [vH02] which detects if a fourth-order equation is a symmetric product. It finds $\mathrm{DE}_z = L_+^z \circledP L_-^z$, where

$$\begin{aligned} L_\pm^z := {}& z(1-4z)(1+4z)^2((1+4z)^2-16y^2z)\,\frac{\mathrm{d}^2}{\mathrm{d}z^2} \\ &+ (8z(48z^2+8z-3)y^2+(1-32z^2)(1+4z)^2)(1+4z)\,\frac{\mathrm{d}}{\mathrm{d}z} \\ &+ (64z^3+80z^2+4z-1)y^2-3z(1+4z)^3 \\ &\pm (1-8z)y\sqrt{2(1-4z)((1+4z)^2-16zy^2)}. \end{aligned} \tag{2.1}$$

For several reasons mentioned in the introduction, we very much expected L_+^z and L_-^z to have ${}_2F_1$ type solutions. Our search for ${}_2F_1$ solutions was time consuming because the algorithms for this are only implemented for rational function coefficients. (The square root in $L_\pm^z$ has z-degree 3, so it cannot be rationalized.)

2.2 Decomposing DE_y

Next we applied algorithm [vH02] to DE_y and found it to be a symmetric product as well, namely $L_+^y \circledP L_-^y$ where

$$\begin{aligned} L_\pm^y = {}& 2(1-y^2)(2y^2-4z-1)^2((1+4z)^2-16zy^2)\,\frac{\mathrm{d}^2}{\mathrm{d}y^2} \\ &+ 4y(2y^2-4z-1)(32zy^4-(1+4z)((1+28z)y^2-4z(3+4z)))\,\frac{\mathrm{d}}{\mathrm{d}y} \\ &+ 16(2y^2-14z-3)zy^4+(1+4z)(1+32z+80z^2)y^2-(1+4z)^2(1+4z+8z^2) \\ &\pm (2(1+8z)y^2-4(1+4z)(1-z))y\sqrt{2(1-4z)((1+4z)^2-16zy^2)}. \end{aligned} \tag{2.2}$$

The polynomial in the square root has y-degree 2. So this time we can rationalize it with a substitution. We choose

$$y = \left(x+(t^2-1)^2\right)\sqrt{z/x}, \quad z = \frac{1}{4(1-2t^2)}. \tag{2.3}$$

Substituting (2.3) in (2.2) produces equations $\tilde{L}_\pm^x$. The factor $\sqrt{z/x}$ in y does not cause a square root in $\tilde{L}_\pm^x$ because $F(y,z) = F(-y,z)$.

If a differential equation has non-zero exponents $e_1 \le e_2$ at a point p, then dividing its solutions by $(x-p)^{e_1}$ creates an exponent 0 at that point, which usually reduces the size of the equation. Applying this idea to $\tilde{L}^x_+$ we divide its solutions by

$$\sqrt[4]{x\frac{x+(t-1)(t+1)^3}{x+(t+1)(t-1)^3}} \tag{2.4}$$

which results in

$$L^x_+ = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{1}{x} - \frac{1}{x+(t+1)(t-1)^3} + \frac{2(x+t^4+2t^2-1)}{x^2+(2t^4+4t^2-2)x+(t^2-1)^4}\right)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{4x^2+(8t^4-16t^3-4t^2+20t-9)x+(4t^4-8t^3+12t^2+4t-5)(t+1)(t-1)^3}{16x(x^2+(2t^4+4t^2-2)x+(t^2-1)^4)(x+(t+1)(t-1)^3)}.$$

Repeating this for L^y_- produces $L^x_- = L^x_+|_{t\mapsto -t}$. Then

$$\mathrm{DE}_y = L^x_1 \circledP L^x_+ \circledP L^x_-$$

where (2.3) expresses y,z in terms of x,t and where $L^x_1 = \frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2x}$ whose solution $\sqrt{x}$ is the product of (2.4) for t and $-t$.

Let $F^x_+ = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Q}(t)[[x]]$ be the unique monic ($a_0 = 1$) non-logarithmic solution of L^x_+ at $x = 0$. Replacing t with $-t$ gives the corresponding solution F^x_- of L^x_-. Then the product

$$\sqrt{x} \cdot F^x_+ \cdot F^x_- \tag{2.5}$$

is a solution of DE_y. The fact that $F(y,z)$ also satisfies DE_y does not directly provide a functional equality: $F^x_\pm$ are series at $x = 0$, which under (2.3) corresponds to $y = \infty$, but that makes the radius of convergence (R_y from Remark 1.1) equal to 0. This problem disappears if we take solutions of $L^x_\pm$ at another singularity, namely $x = -(1+\sqrt{2t^2-1})^4 = -(1+1/(2\sqrt{-z}))^4$ (note that $\sqrt{-z}$ appears in Theorem 1.1). The next task is to solve $L^x_\pm$, which unexpectedly produced a hyperelliptic integral.

2.3 Solving the differential equation with a Hadamard product

To ensure that all ${}_2F_1$ algorithms [vHK19, IvH17] inside `hypergeometricsols` from `Maple` 2021 were applicable, we reduced the coefficients of L^x_+ to $\mathbb{Q}(x)$ by substituting values for t. However, to our surprise, we found no ${}_2F_1$ solution. In an effort to prove this, we computed the monodromy group in Section 4.

Since $F(y,z) \in \mathbb{Z}[y]((z))$, we expect equations obtained from $F(y,z)$ to be arithmetic; we expect that there exists some $c \in \mathbb{Q}(t)\setminus\{0\}$ such that $a_n c^n \in \mathbb{Z}[t]$ for all n. This turned out to be the case with $c = 2^6(t^2-1)^4$. To our surprise — the intermediate equations $L^z_\pm$ and $L^y_\pm$ did not have this feature — not only was $a_n c^n$ in $\mathbb{Z}[t]$, it was even divisible by $\binom{2n}{n}$. Then $\sum \tilde{a}_n x^n$ with $\tilde{a}_n := a_n/\binom{2n}{n}$ should also satisfy an arithmetic equation. One can find this equation with the package `gfun` in `Maple`, which can convert L^x_+ to recurrence for a_n, compute a recurrence for $\tilde{a}_n$, and convert back to a differential equation. By solving that equation we expected to find a ${}_2F_1$ expression for $\sum \tilde{a}_n x^n$ but, surprisingly, this function is algebraic, similar to (2.6) below.

Let $L^m = L^x_+\big|_{x\mapsto m}$ where $m := \frac{1}{64x} - (t+1)(t-1)^3$. Solving L^m is equivalent to solving L^x_+. The Möbius transformation $x \mapsto m$ moves the apparent singularity to $x = \infty$ and reduces expression sizes in equations (2.7), (2.8) below. The exponents at $x=0$ are $\frac{1}{2},\frac{1}{2}$ so L^m has a unique solution of the form $x^{1/2}\sum u_n x^n$ with $u_0=1$. A recurrence for u_n can be computed with `gfun`, it is given in Theorem 2.1 below. Now let $\tilde{u}_n := u_n/\binom{2n}{n}$ and $Z(x) := \sum \tilde{u}_n x^n$. After computing and solving a differential equation for $Z(x)$ we find

$$Z(x) = \frac{R_-^{1/8}+R_+^{1/8}}{2\sqrt{S}} \tag{2.6}$$

where

$$\begin{aligned} R_\pm &= A \pm \sqrt{A^2-1},\\ S &= (1-16(t+1)(t-1)^3x)(1+2^6(t^3+t^2-t)x-2^{10}(t^3+t^2)(t-1)^3x^2), \qquad (2.7)\\ A &= 1+2^7(2t-1)^2x-2^{11}(t-1)^3(2t-1)(2t^2+5t-1)x^2\\ &\quad +2^{17}t(t-1)^6(2t^2+2t-1)x^3-2^{21}(t^3+t^2)(t-1)^9x^4. \qquad (2.8) \end{aligned}$$

For any X the expression $(1+16X)^{1/8}$ is in $\mathbb{Z}[[X]]$. Applying this to $R_\pm^{1/8} = (1\pm 16\sqrt{x}+\cdots)^{1/8}$ we find $R_\pm^{1/8} \in \mathbb{Z}[t][[\sqrt{x}]]$. Since $R_- = R_+|_{\sqrt{x}\mapsto-\sqrt{x}}$, the numerator of (2.6) is in $2\mathbb{Z}[t][[x]]$ and hence $Z(x)\in\mathbb{Z}[t][[x]]$. This establishes an Apéry-like result.

Theorem 2.1. *Define degree $4n$ polynomials u_n by $u_n=0$ for $n<0$, $u_0=1$, and*

$$\begin{aligned} &(n+1)^2u_{n+1}-2^2(16(t^4-6t^3-4t^2+6t-1)(n^2+n)+4t^4-24t^3-12t^2+20t-3)u_n\\ &\quad -2^{11}t(t-1)^3(t+1)(8(t^2+2t-1)n^2-2t^2-6t+3)u_{n-1}\\ &\quad +2^{18}t^2(t-1)^6(t+1)^2(2n+1)(2n-3)u_{n-2}=0 \end{aligned} \tag{2.9}$$

for $n\geq 0$. Then $u_n\in\mathbb{Z}[t]$, and is divisible by $\binom{2n}{n}$.

Similar results are found by computing recurrences for solutions of L^m or $L^x_\pm$ at other singular points. The smallest recurrence we found this way is (2.9).

The generating function of $\binom{2n}{n}$ is $1/\sqrt{1-4x}$, so we can express $\sum u_n x^n$ (the solution of L^m without the factor $x^{1/2}$) as

$$\sum_{n=0}^\infty u_n x^n = \sum_{n=0}^\infty \binom{2n}{n}\tilde{u}_n x^n = \frac{1}{\sqrt{1-4x}} \star Z(x),$$

where the Hadamard product $\star$ is given by $\sum a_n x^n \star \sum b_n x^n = \sum a_n b_n x^n$.

2.4 Reducing the Hadamard product to a hyperelliptic integral

In general, if $u_n=\binom{2n}{n}\tilde{u}_n$ and $Z(x)=\sum_{n=0}^\infty \tilde{u}_n x^n$ then

$$\sum_{n=0}^\infty u_n x^n = \frac{1}{\sqrt{1-4x}}\star Z(x) = \frac{1}{\pi}\int_0^{4x}\frac{Z(\xi)}{\sqrt{\xi(4x-\xi)}}\,\mathrm{d}\xi. \tag{2.10}$$

Now let $Z(x)$ be the algebraic function (2.6). The algebraic relation between $Z(x)$ and x is

$$\frac{((4SZ^2-2)^2-2)^2-2-2A}{1-16(1+t)(1-t)^3x} = \text{an explicit but lengthy polynomial in } t,x,Z.$$

Remarkably, there is a rational parametrization, however, allowing a square root makes the parametrization *much* shorter:

$$x = \frac{(t^2 - v^2)}{16t(tv^4 + (t+1)(t-1)^2(t^2 - 2v^2 - t))}, \tag{2.11}$$

$$Z = \frac{(t-1+v)(tv^4 + (t+1)(t-1)^2(t^2 - 2v^2 - t))}{v(v^4 - 2t^2v^2 + (t^2-1)^2)} \sqrt{\frac{v-t}{2t(t^2+tv-1)}}. \tag{2.12}$$

Substituting the right-hand sides of (2.12), (2.11) for $Z(\xi), \xi$ in (2.10) gives

$$\frac{\sqrt{-2}}{\pi} \int \frac{(v+t-1)\,\mathrm{d}v}{\sqrt{(v+t)(v+t-t^{-1})(v^2 - t^2 + 64tx(tv^4 + (t+1)(t-1)^2(t^2 - 2v^2 - t)))}}. \tag{2.13}$$

Reinserting the factor $x^{1/2}$ and reversing the transformation $x \mapsto m$ gives a solution of L_+^x. Replacing t by $-t$ gives a solution of L_-^x. Then we get a solution of DE_y just like (2.5). Reversing (2.3) to write it in terms of y and z produces a formula similar to Theorem 1.1. We made further substitutions for u and x to minimize the expression size, see our website [vH23] for details. To verify that the resulting formula, in Theorem 1.1, is still correct, we computed a differential equation for it and compared initial conditions.

3 Curves with real multiplication

Let $\mathcal{L} = \mathbb{Q}(u, \sqrt{2u^2 - 2u})$ where $u = 4z$ and $\sqrt{2u^2 - 2u}$ appear in Theorem 1.1. Parametrization (2.3) rationalizes this to $\sqrt{2u^2 - 2u} = 2tu$ and $u = 1/(1-2t^2)$, so $\mathcal{L} = \mathbb{Q}(t)$. One of the most remarkable aspects is that the hyperelliptic integrals $I_\pm(u,x)$ from Theorem 1.1 satisfy second-order equations, namely

$$L_\pm^{\mathrm{PF}} := L_\pm^x\big|_{x \mapsto \frac{-x}{4u^2}} \in \mathcal{L}(x)[\mathrm{d}/\mathrm{d}x]. \tag{3.1}$$

The Picard–Fuchs equation for a hyperelliptic integral of genus g almost always has order $2g$, so it is very surprising that $I_\pm(u,x)$ satisfies an equation of order 2. This motivates us to look for special properties of the corresponding hyperelliptic curve (3.2). We detected multiplication by $\sqrt{2}$ in our monodromy computation in Section 4. Here we verify this property with a Humbert relation.

Theorem 1.1 contains the square root of

$$H := v(1-v)\left((1-v)(1-u^2v)(1+uv)^2 + xv(1-uv)^2\right) \in \mathbb{Q}(u,x)[v]. \tag{3.2}$$

The equation $Y^2 = H(x,u,v)$ defines a family of hyperelliptic curves $C = C^{u,x}$. Special properties may be visible in the Jacobian $\mathrm{Jac}(C) := H^0(\Omega_C)^*/H_1(C,\mathbb{Z})$ which is isomorphic to a two-dimensional complex torus $\mathbb{C}^2/\Lambda$. To obtain Λ, choose a symplectic basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ (Figure 4.1 in Section 4) for the homology group $H_1(C,\mathbb{Z})$, and a basis of holomorphic differentials:

$$\omega_1 := \frac{\mathrm{d}v}{\sqrt{H}}, \quad \omega_2 := \frac{v\,\mathrm{d}v}{\sqrt{H}} \in H^0(\Omega_C). \tag{3.3}$$

Now Λ, the period lattice, is generated over $\mathbb{Z}$ by the columns of the period matrix

$$\begin{pmatrix} \int_{\alpha_1}\omega_1 & \int_{\alpha_2}\omega_1 & \int_{\beta_1}\omega_1 & \int_{\beta_2}\omega_1 \\ \int_{\alpha_1}\omega_2 & \int_{\alpha_2}\omega_2 & \int_{\beta_1}\omega_2 & \int_{\beta_2}\omega_2 \end{pmatrix}. \tag{3.4}$$

After updating the basis ω_1, ω_2 this becomes

$$\begin{pmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & 1 & \tau_2 & \tau_3 \end{pmatrix}.$$

Now τ_1, τ_2, τ_3 only depend on the chosen symplectic basis.

If the Jacobian has non-trivial endomorphisms, this is reflected in a *Humbert* relation between τ_1, τ_2 and τ_3, which is of the form

$$A\tau_1 + B\tau_2 + C\tau_3 + D(\tau_2^2 - \tau_1\tau_3) + E = 0$$

for some $A,B,C,D,E \in \mathbb{Z}$. Inside the three-dimensional moduli space $\mathcal{M}_3$ of all curves of genus 2 or, what amounts to the same, the space of $\mathcal{A}_2$ of (principally polarised) abelian surfaces, the curves that satisfy such a Humbert relation form a *Humbert surface* H_Δ of discriminant

$$\Delta := B^2 - 4AC - 4DE.$$

For a general curve belonging to H_Δ the endomorphisms of the Jacobian form an order in the real quadratic field with discriminant Δ, and we say that the curve (or its Jacobian) has *real multiplication* by the order.

By numerical evaluation of the period matrix, after picking numerical values for u, x, one can check that our curves $C^{u,x}$ belong to the Humbert surface H_8:

$$\mathbb{Z}[\sqrt{2}] \subseteq \mathrm{End}(\mathrm{Jac}(C^{u,x})).$$

However, one can do better than approximate verification. The insight of Humbert [Hum99] was that, for each Δ, there is a polynomial condition on the coefficients a_1, a_2, a_3, a_4 of the genus 2 curve

$$Y^2 = v(v-a_1)(v-a_2)(v-a_3)(v-a_4)$$

to be on the Humbert surface H_Δ. For $\Delta = 8$ the condition, already found by Humbert [Hum99, p. 327], reads

$$4a_1a_2a_3a_4((a_1+a_3)(a_2+a_4) - 2a_1a_3 - 2a_2a_4)^2 = (a_2-a_4)^2(a_1-a_3)^2(a_1a_3+a_2a_4)^2.$$

Using this, it is trivial to verify the following statement.

Theorem 3.1. *The smooth curves $C^{u,x}$ admit real multiplication by $\sqrt{2}$. In other words, the endomorphism ring of their Jacobian contains $\mathbb{Z}[\sqrt{2}]$.*

4 Monodromy calculation

We will now study in some more detail the geometry of the curves $C^{u,x}$. For each value of u, $C^{u,x}$ becomes a 1-parameter family of curves, parametrized by x, forming a surface S^u. The singular members in this family are

$$x_1 = u^2 - 6u + 1 + 4(1-u)\sqrt{-u}, \quad x_2 = 0, \quad x_3 = u^2 - 6u + 1 - 4(1-u)\sqrt{-u}, \quad x_4 = \infty.$$

These are also the non-apparent singularities of the Picard–Fuchs equation L^{PF}, the minimal differential equation in $\mathcal{L}(x)[\mathrm{d}/\mathrm{d}x]$ for the periods in (3.4). It has the expected order $2g = 4$. Our aim is to find its monodromy representation.

As the local system of solutions to L^{PF} can be identified with the local system of homologies $H_1(C^{u,x}, \mathbb{Z})$ tensored with $\mathbb{C}$, this explicit form can be obtained by elementary topological means. To achieve this, we temporarily fix the value $u = 1/2$. Then the singularities are

$$x_1 = -7/4 + \sqrt{-2}, \quad x_2 = 0, \quad x_3 = -7/4 - \sqrt{-2}, \quad x_4 = \infty.$$

We chose a base point

$$x_{\mathrm{BP}} = 1$$

because all roots of H are real at $u = 1/2$, $x = 1$. Sorted $v_1 < \cdots < v_6$ the roots are

$$v_1 = -3 - \sqrt{17}, \;\; v_2 = \frac{3 - \sqrt{17}}{2}, \;\; v_3 = 0, \;\; v_4 = 1, \;\; v_5 = -3 + \sqrt{17}, \;\; v_6 = \frac{3 + \sqrt{17}}{2}.$$

To visualize cycles on the hyperelliptic curve $C^{1/2,1}$ we use classical branch-cuts: in Figure 4.1 we drew oriented cycles $\alpha_1, \beta_1, \alpha_2, \beta_2 \in H_1(C^{1/2,1})$ in the v-plane. When crossing a branch-cut we change the sheet, and continue the cycle by drawing it as a dashed line. We chose branch-cuts running from v_1 to v_2, from v_3 to v_4 and from v_5 to v_6.

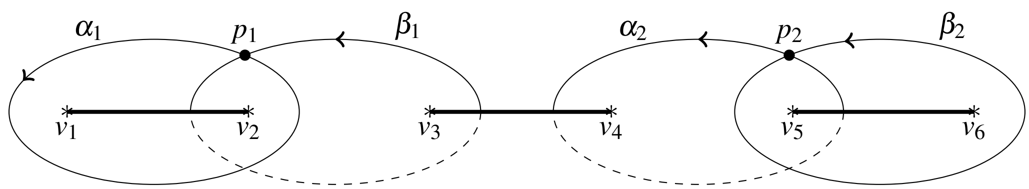

Figure 4.1: A symplectic homology basis $\alpha_1, \beta_1, \alpha_2, \beta_2 \in H_1(C^{1/2,1})$.

Here the cycle α_1 runs counter-clockwise around the cut $v_1 v_2$. It is completely on the 'upper sheet'. The cycle β_1 encircles v_2 and v_3 counter-clockwise. This cycle hits the cuts $v_1 v_2$ and $v_3 v_4$, so when drawing it, the lower part of the cycles is dashed and runs on the 'lower sheet' of the double cover. The cycles α_1 and β_1 intersect in a single point p_1. One has

$$1 = \alpha_1 \cdot \beta_1 = -\beta_1 \cdot \alpha_1,$$

as the tangent to α_1 at the intersection points p_1 has to turn counter-clockwise towards the tangent of β_1 at p_1. Similarly, we take cycles α_2 and β_2, where α_2 encircles v_4 and v_5 (so hits two cuts and has lower half-dashed) and β_2, which encircles the cut $v_5 v_6$, always counter-clockwise. Thus we also have

$$1 = \alpha_2 \cdot \beta_2 = -\beta_2 \cdot \alpha_2.$$

All other intersection products between the cycles are zero, so that $\alpha_1, \beta_1, \alpha_2, \beta_2$ make up a symplectic basis for the homology group $H_1(C^{1/2,1}, \mathbb{Z})$.

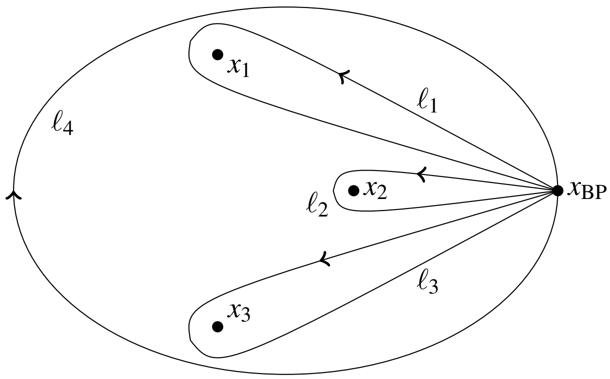

Figure 4.2: Loops around the singularities for $u = 1/2$

Next we examine how continuously changing x (while fixing $u = 1/2$) impacts Figure 4.1. We choose loops $\ell_1, \ell_2, \ell_3, \ell_4$ in the x-plane that start at $x_{\mathrm{BP}} = 1$ and loop around $x_1 = -7/4 + \sqrt{-2}$, $x_2 = 0$, $x_3 = -7/4 - \sqrt{-2}$, $x_4 = \infty$ respectively, see Figure 4.2. Following such a loop, the points $v_1, \dots, v_6$ in Figure 4.1 move, dragging the paths $\alpha_1, \beta_1, \alpha_2, \beta_2$ along with them. When we return to x_{BP} at the end of the loop, we have a permutation of the original roots $v_1, \dots, v_6$, and the dragged paths form a new symplectic basis. The change of basis matrix, between the new and the original basis, is a symplectic matrix $M_i \in \mathrm{Sp}_4(\mathbb{Z}) \subset \mathrm{GL}_4(\mathbb{Z})$ for each ℓ_i. This gives a homomorphism (the *monodromy representation*)

$$\rho\colon \pi_1(\mathbb{P}^1 \setminus \{x_1, x_2, x_3, x_4\}, x_{\mathrm{BP}}) \to \mathrm{Sp}_4(\mathbb{Z}) \cong \mathrm{Aut}(H_1(C^{1/2,1}, \mathbb{Z})),$$

with $\rho(\ell_i) = M_i$. As the composed path $\ell_1\ell_2\ell_3\ell_4$ is contractible to a point we have

$$M_1 M_2 M_3 M_4 = \mathrm{Id}. \tag{4.1}$$

Proposition 4.1. *With the basis for $H_1(C^{1/2,1}, \mathbb{Z})$ in Figure 4.1 and generators $\ell_1, \dots, \ell_4$ for the group $\pi_1(\mathbb{C} \setminus \{x_1, x_2, x_3, x_4\}, x_{\mathrm{BP}})$ in Figure 4.2 we have*[4]

$$M_1 := \begin{pmatrix} 0 & -1 & 1 & 1 \\ 2 & 2 & -1 & -2 \\ 2 & 1 & 0 & -2 \\ -1 & -1 & 1 & 2 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3 := \begin{pmatrix} 2 & -1 & 1 & -1 \\ 2 & 0 & 1 & -2 \\ 2 & -1 & 2 & -2 \\ 1 & -1 & 1 & 0 \end{pmatrix}, \qquad M_4 := \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

[4]This monodromy is close to the type that is classified in [BR13, Theorem 3.1], with a subtle difference. The monodromy in [BR13] is irreducible over $\mathbb{C}$, whereas the monodromy here is irreducible over $\mathbb{Q}$ but reducible over $\mathbb{C}$, see Section 4.1. This subtle difference is important because reducibility over $\mathbb{C}$ is what makes it possible for a hyperelliptic integral to satisfy a differential equation of order 2 instead of the expected order 4. It also causes the real multiplication discussed in Section 3.

Proof. To compute these monodromy matrices we use Picard–Lefschetz theory [HZ77], [Vas02], which tells us that M_i is determined by vanishing cycles. In general, when a curve acquires an A_1-singularity at a certain value of the parameter, there is a *vanishing cycle* δ, that contracts to a point. The monodromy is geometrically a Dehn twist along this cycle. On the level of homology, the monodromy transformation T is a *symplectic reflection* in δ:

$$T_\delta\colon\, \gamma \mapsto \gamma - (\delta\cdot\gamma)\delta. \tag{4.2}$$

A `Maple`-animation showed that if x follows loop ℓ_1 around x_1 then the roots $v_1,\dots,v_6$ of H move as indicated in Figure 4.3. There are *two* A_1-singularities at $x = x_1$, and two disjoint *vanishing cycles* δ_1, δ_2. In such cases, the transformations T_{δ_1} and T_{δ_2} commute. We will compute them next; the monodromy will be their product.

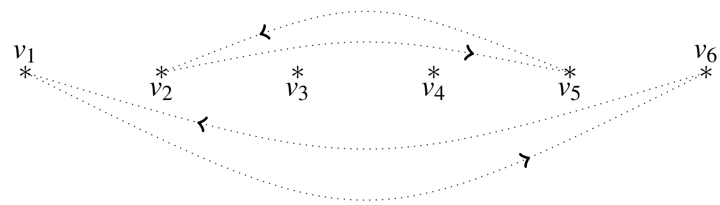

Figure 4.3: Motion of $v_1,\dots,v_6$ when x follows ℓ_1

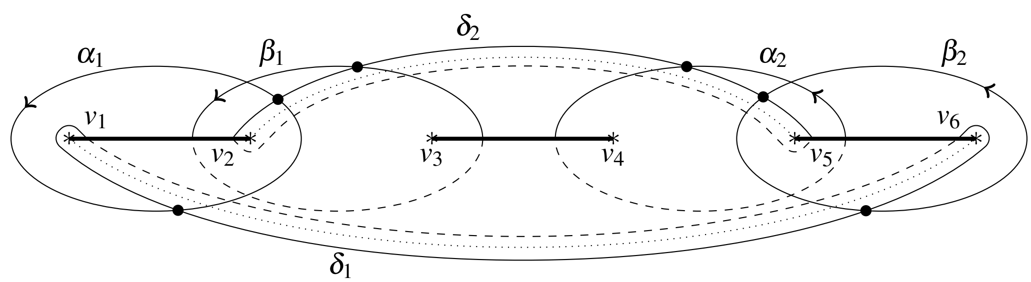

Figure 4.4: Vanishing cycles when x follows ℓ_1

Figure 4.3 shows v_1 and v_6 interchanging in the lower part of the plane, in a *counter-clockwise*[5] manner (i.e., the point that travels left moves *over* the point that travels right). Drawing an auxiliary arc in the lower part of the plane, connecting v_1 and v_6, gives the vanishing cycle δ_1 shown in Figure 4.4. It encircles v_1 and v_6 and is half-dashed, as it intersects the cuts v_1v_2 and v_5v_6. Figure 4.3 also shows v_2 and v_5 interchanging in the upper part of the plane. That corresponds to the vanishing cycle δ_2 in Figure 4.4.

[5] All branch-point swaps in this paper happen to be counter-clockwise. (For clockwise swaps, replace the minus sign in (4.2) by a plus sign, this inverts T_δ.) Unlike swaps, a vanishing cycle δ does not have a natural orientation. Indeed, $T_\delta = T_{-\delta}$ so a clockwise treatment of δ produces the same monodromy as our counter-clockwise treatment.

Careful comparison shows that $\delta_1 = -(\beta_1+\alpha_2)$ in the homology. To ensure correctness, we prove this formula by checking that $\delta_1\cdot\gamma = -(\beta_1+\alpha_2)\cdot\gamma$ for each γ in our basis $\alpha_1,\beta_1,\alpha_2,\beta_2$. Next we apply

$$T_{\delta_1}\colon \gamma\mapsto \gamma-(\delta_1\cdot\gamma)\delta_1$$

to each element in our basis, resulting in the matrix

$$T_{\delta_1} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & -1\\ 1 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Similarly we find and prove $\delta_2 = -\alpha_1+\beta_1+\alpha_2-\beta_2$ with the corresponding symplectic reflection

$$T_{\delta_2} = \begin{pmatrix} 0 & -1 & 1 & 1\\ 1 & 2 & -1 & -1\\ 1 & 1 & 0 & -1\\ -1 & -1 & 1 & 2\end{pmatrix}.$$

We verify that T_{δ_1} and T_{δ_2} commute, as they should, and obtain the monodromy by taking their product

$$M_1 := T_{\delta_1}T_{\delta_2} = \begin{pmatrix} 0 & -1 & 1 & 1\\ 2 & 2 & -1 & -2\\ 2 & 1 & 0 & -2\\ -1 & -1 & 1 & 2\end{pmatrix}.$$

The first step towards finding M_2 is Figure 4.5. This time v_1 and v_2 change place, with v_2 going *over* v_1. At the same time v_5 makes a complete counter-clockwise turn around v_4.[6] The auxiliary arc connecting v_1 and v_2 can be taken to be the branch-cut v_1v_2, so the vanishing cycle is simply α_1.

The matrix of the map $T_{\alpha_1}\colon \gamma\mapsto\gamma-(\alpha_1\cdot\gamma)\alpha_1$ is

$$T_{\alpha_1} = \begin{pmatrix} 1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

The loop of v_5 around v_4 can be seen as the *square* of the interchanging v_4,v_5 — this is just the square of T_{α_2}(!) with

$$T_{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

[6]The degeneration of $C^{u,x}$ when x approaches x_2 is slightly different from the other cases. The local monodromy around x_2 is the product of a symplectic reflection and the *square* of another (commuting with the first) symplectic reflection. This leads then to a coefficient 2 in the Jordan form. For $x=x_2=0$ the two double factors appearing in the equation $(v-1)^2$ and $(uv+1)^2$ behave differently: the point $(x=0,y=0,v=1)$ is a *singular* point of the surface S^u, whereas the point $(x=0,y=0,v=-1/u)$ is a non-singular point. In local coordinates putting the singularity at the origin, the first point has normal form $y^2=v^2+x^2$ (singularity of the surface), whereas the second $y^2=v^2+x$ (smooth point on the surface). Because we have x^2 in the first normal form, the local monodromy is the *square* of a symplectic reflection.

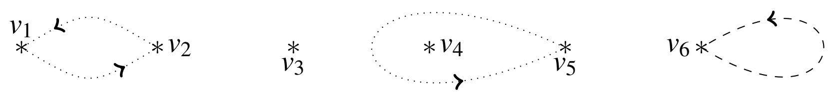

Figure 4.5: Motion of $v_1, \ldots, v_6$ when x follows ℓ_2

Thus the monodromy M_2 corresponding to the loop ℓ_2 is

$$M_2 := T_{\alpha_1} T_{\alpha_2}^2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Loop ℓ_3 is very similar to loop ℓ_1, the motion of $v_1, \ldots, v_6$ differs in a subtle way, as shown in Figure 4.6. There are again two A_1-singularities, and two disjoint vanishing cycles δ_3, δ_4.

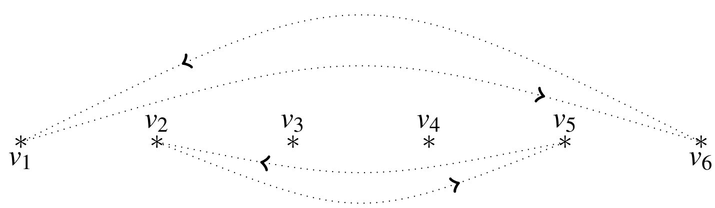

Figure 4.6: Motion of $v_1, \ldots, v_6$ when x follows ℓ_3

Figure 4.6 shows v_1 and v_6 changing place. They move in the upper part of the plane and v_6 moves *over* v_1. At the same time, v_2 and v_5 change place as well (again, v_5 moves *over* v_2). If x approaches the x_3, then v_1 and v_6 coalesce. Now draw an auxiliary arc in the upper part of the plane, connecting v_1 and v_6. The vanishing cycle δ_4 is then the cycle that encircles this arc (counter-clockwise) at small distance (see Figure 4.7).

The vanishing cycle δ_3 comes from the coalescence of v_2 and v_5 at the singularity. We take an auxiliary arc, connecting v_2 and v_5, now in the lower part of the plane, and δ_3 runs at close distance counter-clockwise around this arc. Like before, we find and prove $\delta_3 = \alpha_1 + \beta_1 + \alpha_2 + \beta_2$. We apply $T_{\delta_3} \colon \gamma \mapsto \gamma - (\delta_3 \cdot \gamma)\delta_3$ to each element of our basis, resulting in the matrix

$$T_{\delta_3} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

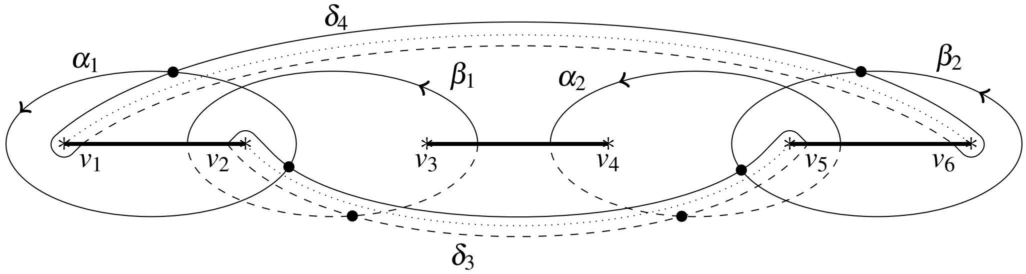

Figure 4.7: Vanishing cycles when x follows ℓ_3

Since $\delta_4 = \beta_1 + \alpha_2$ equals $-\delta_1$ from ℓ_1 we get $T_{\delta_4} = T_{\delta_1}$. We verified that T_{δ_3} and T_{δ_4} commute, as they should. Their product is:

$$M_3 := T_{\delta_3} T_{\delta_4} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 2 & 0 & 1 & -2 \\ 2 & -1 & 2 & -2 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

Finally, $M_4 := (M_1 M_2 M_3)^{-1}$ has the expected Jordan-structure consisting of two 2×2 blocks, like matrices M_1, M_2, M_3, except that the eigenvalues are -1 instead of 1. To guard against mistakes, we independently computed each matrix with numerical analytic continuation, and found the same $M_1, \ldots, M_4$ to high precision [vH23]. $\square$

4.1 Monodromy and real multiplication

With $\mathcal{L}$ as in Section 3, the integrals from Theorem 1.1 are

$$I_\pm(u,x) = \frac{1}{\pi} \int_0^1 \omega_1 + c_\pm \omega_2,$$

where $c_\pm = -u \pm \sqrt{2u^2 - 2u} \in \mathcal{L}$ are linear combinations of the periods, and hence they are solutions of L^{PF} (from Section 4). Indeed,

$$L^{\mathrm{PF}} = \mathrm{LCLM}(L_+^{\mathrm{PF}}, L_-^{\mathrm{PF}}) \in \mathcal{L}(x)[\mathrm{d}/\mathrm{d}x]. \tag{4.3}$$

The remarkable feature that our hyperelliptic integrals $I_\pm(u,x)$ manage to satisfy second-order equations, instead of the expected $2g = 4$, corresponds to the fact that the fourth-order Picard–Fuchs operator L^{PF} decomposes into two second-order operators (equation (4.3)). This in turn is equivalent to $\mathbb{C}^4$ decomposing as a direct sum of 2-dimensional G-invariant subspaces, where $G := \langle M_1, M_2, M_3 \rangle$ is the *monodromy group* of L^{PF}. It is not *a priori* obvious that this property is possible for a family of hyperelliptic curves.

The two G-invariant subspaces are defined over $\mathbb{Q}(\sqrt{2})$. That suggests that $C = C^{u,x}$ has real multiplication by $\sqrt{2}$, i.e., $\mathbb{Z}[\sqrt{2}] \subseteq \mathrm{End}(\mathrm{Jac}(C))$, as was verified in Section 3. Endomorphisms of $\mathrm{Jac}(C)$

induce endomorphisms of the lattice $H_1(C,\mathbb{Z}) = \mathbb{Z}^4$ that commute with the monodromy representation

$$\rho\colon \pi_1(\mathbb{C}\setminus\{x_1,x_2,x_3,x_4\},x_{\mathrm{BP}}) \to \mathrm{Sp}_4(\mathbb{Z}),$$

where $\rho(\ell_i) = M_i$.

For a representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{Z})$, the endomorphism ring $\mathrm{End}(\rho)$ consists of all matrices $A \in \mathrm{GL}_n(\mathbb{Z})$ such that $\rho(g)A = A\rho(g)$ holds for all $g \in G$. Direct computation with the matrices from Proposition 4.1 shows that our ρ has the following endomorphism ring.

Proposition 4.2. $\mathrm{End}(\rho) = \mathbb{Z}\,\mathrm{Id} \oplus \mathbb{Z}\,\mathrm{RM} \cong \mathbb{Z}[\sqrt{2}]$ *where*

$$\mathrm{RM} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is a matrix with $\mathrm{RM}^2 = 2\,\mathrm{Id}$.

By taking bases of the G-invariant subspaces of $\mathbb{C}^4$ (or by taking eigenvectors of RM) we obtain the columns of the matrix

$$P := \begin{pmatrix} 0 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ \sqrt{2} & 0 & -\sqrt{2} & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Consequently, the matrices $P^{-1}M_iP$ have block diagonal form

$$P^{-1}M_iP = \begin{pmatrix} A_i & 0 \\ 0 & \tilde{A}_i \end{pmatrix}. \tag{4.4}$$

We find

$$A_1 := \begin{pmatrix} 2-\sqrt{2} & -\frac{1}{2}\sqrt{2}+1 \\ -2+\sqrt{2} & \sqrt{2} \end{pmatrix}, \quad A_2 := \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$
$$A_3 := \begin{pmatrix} -\sqrt{2} & \frac{1}{2}\sqrt{2}+1 \\ -2-\sqrt{2} & 2+\sqrt{2} \end{pmatrix}, \qquad A_4 := \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

The matrices $\tilde{A}_i$ are obtained from A_i with the Galois automorphism $\sqrt{2} \mapsto -\sqrt{2}$. A splitting like (4.4) is equivalent to the decomposition of L^{PF} as the LCLM of two second-order equations. It is interesting that the real multiplication of the curves is visible in $\mathrm{End}(\rho)$.

Theorem 4.1. *The monodromy group of* $L_{\pm}^{\mathrm{PF}}$ *is a* dense *subgroup of* $\mathrm{SL}_2(\mathbb{R})$.

Proof. The group $\mathrm{SL}_2(\mathbb{R})$ is generated by

$$U := \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\} \quad \text{and} \quad L := \left\{ \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} : r \in \mathbb{R} \right\}.$$

A computer search for 'small' elements of the monodromy group $\langle A_1, A_2, A_3\rangle$ produced

$$u = \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad l = \begin{pmatrix} 1 & 0 \\ 4\sqrt{2} & 1 \end{pmatrix}.$$

Now $\langle u, A_4^2\rangle$ is a dense subgroup of U, while $\langle l, A_2\rangle$ is a dense subgroup of L. □

Corollary 4.1. *If $L_{\pm}^{\mathrm{PF}}$ is solvable in terms of a ${}_2F_1$ function, then the monodromy group of the Euler–Gauss hypergeometric equation for this ${}_2F_1$ must have a finite-index subgroup that is defined over $\mathbb{Q}(\sqrt{2})$ and is dense in* $\mathrm{SL}_2(\mathbb{R})$.

Our goal is to establish that $L_{\pm}^{\mathrm{PF}}$ is not ${}_2F_1$-solvable, and it appears that we are *almost* done. All we have to do is to classify the ${}_2F_1(a,b;c \mid x)$ that satisfy the assumption in Corollary 4.1, and show that the list is empty. However, it is not! The monodromy of ${}_2F_1(\frac14, \frac58; 1 \mid x)$ satisfies this assumption. And so the saga continues.

4.2 Algebraic automorphisms

How can we rule out a solution in terms of ${}_2F_1(\frac14, \frac58; 1 \mid f)$ for some algebraic function f? Call $d := [\mathbb{C}(f,x) : \mathbb{C}(x)]$ the *algebraic degree* of f. An example with $d = 2$ that comes from modular forms appears in the ${}_2F_1$-expression for the generating function of the Apéry sequence. The hypergeometric series ${}_2F_1(\frac14, \frac58; 1 \mid x)$ is of a different nature, its Schwarz triangle has angles (after scaling down by π) $|1-c| = 0$, $|c-a-b| = \frac18$ and $|b-a| = \frac38$, so it is not even on Takeuchi's list of arithmetic triangle groups [Tak77]. However, there do exist examples with $d > 1$ that do not come from modular forms, for instance:

$$2x(25x^2+14x+25)y'' + (75x^2+28x+25)y' + (3x+\tfrac12)y = 0. \tag{4.5}$$

Here $d = 2$, which means that it is not ${}_2F_1$-solvable with a rational pullback, but it does have a ${}_2F_1$-solution with a pullback f in a quadratic extension. The implementation from [IvH17] finds a solution with $f \in \mathbb{Q}(x, \sqrt{25x^3+14x^2+25x})$.

With our techniques for finding ${}_2F_1$-type solutions we rule out $d = 1$ for $L_{\pm}^{\mathrm{PF}}$, and $d = 2$; the main difficulty is excluding a ${}_2F_1$-type solution without knowing a bound on d. We will outline a strategy to potentially reduce to the case $d = 1$.

If a second-order differential equation can be solved in terms of ${}_2F_1(a,b;c \mid f)$ for two different pullback functions f, then we say that ${}_2F_1(a,b;c \mid f)$ has *non-unique pullbacks*. If the equation has rational function coefficients but the algebraic degree is not 1, then taking two conjugates $f_1 \neq f_2$ of f means that the ${}_2F_1$ has non-unique pullbacks. If $g \in \overline{\mathbb{C}(x)}$ is the composition of f_1 with the inverse of f_2, then then for a suitable algebraic pre-factor r, the functions ${}_2F_1(a,b;c \mid x)$ and $r \cdot {}_2F_1(a,b;c \mid g)$ will satisfy the same Euler–Gauss hypergeometric equation. We call such $g \in \overline{\mathbb{C}(x)}$ with $g \neq x$ an *algebraic automorphism*. Non-unique pullbacks $f_1 \neq f_2$ can occur even in non-modular and non-arithmetic situations, like

$${}_2F_1\left(\begin{matrix} \frac{1}{12}, \frac16 \\ \frac12 \end{matrix} \,\middle|\, f_1(x)\right) = (1+x)^{-1/2} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{12}, \frac16 \\ \frac12 \end{matrix} \,\middle|\, f_2(x)\right),$$

$$\text{with} \quad f_1(x) = -x(1+x)(3+4x+4x^2)^2 \ \text{ and } \ f_2(x) = f_1\left(\frac{-x}{1+x}\right) = \frac{x(3+2x+3x^2)^2}{(1+x)^6},$$

and

$$
{}_2F_1\left(\begin{matrix}\frac{1}{40},\frac{11}{40}\\ \frac{4}{5}\end{matrix}\,\middle|\, f_1(x)\right) = \left(1-\frac{44x}{25}\right)^{-1/10} \cdot {}_2F_1\left(\begin{matrix}\frac{1}{40},\frac{11}{40}\\ \frac{4}{5}\end{matrix}\,\middle|\, f_2(x)\right),
$$

$$
\text{with} \quad f_1(x) = \frac{256x(1-x)^5}{25-44x+20x^2}
$$

$$
\text{and} \ \ f_2(x) = f_1\left(\frac{-x}{1-\frac{44}{25}x}\right) = -\frac{256x(25-19x)^5}{25(25-44x)^4(25-44x+20x^2)}.
$$

Here f_1, f_2 are rational, but algebraic automorphisms do not always correspond to genus 0 extensions of $\mathbb{C}(x)$, e.g., in the case of (4.5). So far, all non-trivial cases[7] correspond to Schwarz triangles with all angles either 0 and/or reciprocals of a positive integer, which does not include ${}_2F_1(\frac{1}{4}, \frac{5}{8}; 1 \mid x)$.

Another way to potentially prove that ${}_2F_1(\frac{1}{4}, \frac{5}{8}; 1 \mid x)$ has no algebraic automorphisms is by showing that a series expansion of any candidate g has infinitely many primes in its denominators, which implies that g is not algebraic. There is a formula for the power series g in terms of its first term [vH13]. The formula includes integration, which divides the coefficient of x^n by $n+1$. When $n+1$ is prime p, the division introduces p in the denominator, unless a certain congruence $c_p \equiv 1 \bmod p$ holds, where c_p is defined below. The algebraicity of g then implies that congruence for all sufficiently large primes p. Thus, if we could prove that the congruence does not hold for infinitely many primes, it contradicts g being an *algebraic* automorphism. Computing c_p for a number of primes leads to the following conjecture.

Conjecture 4.1. Define

$$
Y(x) = \frac{9}{16}(1-x)^{7/8} \cdot {}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{5}{8}\\ 1\end{matrix}\,\middle|\, x\right)^2,
$$

and let c_n be the coefficient of x^n in $1/Y(x)$. Let $p > 3$ be a prime number. Then

$$
c_p \equiv 1 \bmod p \iff p \equiv \pm 1 \bmod 8.
$$

In particular, $c_p \not\equiv 1 \bmod p$ for infinitely many primes.

Repeating this for other ${}_2F_1$ functions leads to other similar conjectures, such as:

Conjecture 4.2 (exponent-differences 0, $1/2$, $1/k$)**.** For $k \geq 3$, define

$$
Y(x) = \frac{3k^2+4}{8k^2}(1-x)^{1/2} \cdot {}_2F_1\left(\frac{1}{4}-\frac{1}{2k}, \frac{1}{4}+\frac{1}{2k}; 1 \,\middle|\, x\right)^2.
$$

Let c_n be the coefficient of x^n in the series expansion of $1/Y(x)$ at $x = 0$. Then for all but finitely many primes p,

$$
c_p \equiv 1 \bmod p \iff p \equiv \pm 1 \bmod k.
$$

Conjecture 4.2 is true for $k = 3, 4, 6$. (For these k, the ${}_2F_1$ does have algebraic automorphisms, which implies the congruence for all but finitely many primes p.)

[7] If two of the exponent-differences $|1-c|$, $|c-a-b|$, $|b-a|$ are equal, then a Möbius transformation that swaps the two points trivially gives an 'algebraic' automorphism $g \in \mathbb{Q}(x)$.

To summarise: while we cannot prove Conjecture 4.1, it is easy to test it for thousands of primes. So it is exceedingly unlikely that ${}_2F_1(\frac{1}{4},\frac{5}{8};1\mid x)$ has an algebraic automorphism. If this were proven, it would imply unique — hence rational — pullbacks, allowing us to conclude that $L_{\pm}^{\mathrm{PF}}$ is not ${}_2F_1$-solvable.

Finally, note that a description (and understanding) of all ${}_2F_1$ cases with algebraic automorphisms is an interesting question itself.

4.3 Braid action

We computed the monodromy of L^{PF} in Section 4 for one value of u, namely $u=1/2$. As long as the paths in diagrams do not change, the monodromy matrices $M_1,\dots,M_4$ in Proposition 4.1 depend continuously on u. But they have integer entries, so they must be *locally constant*. The monodromy matrices $A_1,\dots,A_4$ of L_+^{PF} were derived from $M_1,\dots,M_4$ in Section 4.1, so they must be locally constant as well.

However, a sufficiently large change in u does alter paths. Figure 4.8 shows what can happen when u moves away from $1/2$, with x_1 and x_3 moving as a result, while $x_2=0$ and $x_4=\infty$ are fixed.

For M_i to depend continuously on u (and thus remain constant) the path ℓ_i around x_i must be dragged along. Figure 4.8 shows that when u moves far enough, a dragged path need no longer be the shortest path around x_i. If we recompute the M_i's using the shortest paths ℓ_i^{new}, and compare with the original $u=1/2$ monodromy matrices belonging to the dragged paths ℓ_i, then the difference corresponds to a *braid action*. Figure 4.8 shows how the monodromy over ℓ_1, ℓ_2, ℓ_3 determines the monodromy over the new paths ℓ_1^{new}, ℓ_2^{new}, ℓ_3^{new}. Since $\ell_1^{\mathrm{new}}=\ell_2$, its matrix M_1^{new} will be M_2. In the figure, $\ell_3^{\mathrm{new}}=\ell_3$ and $\ell_4^{\mathrm{new}}=\ell_4$ (not drawn), hence $M_3^{\mathrm{new}},M_4^{\mathrm{new}}$ are the same as M_3,M_4. Then from (4.1) we find

$$M_1M_2M_3M_4=\mathrm{Id}=M_1^{\mathrm{new}}M_2^{\mathrm{new}}M_3^{\mathrm{new}}M_4^{\mathrm{new}}=M_2M_2^{\mathrm{new}}M_3M_4$$

and hence $M_2^{\mathrm{new}}=M_2^{-1}M_1M_2$. This procedure gives a map

$$\mathrm{Braid}_{1,2}\colon\ \mathrm{GL}_4(\mathbb{C})^4\to\mathrm{GL}_4(\mathbb{C})^4$$

which sends $(M_1,\dots,M_4)$ to $(M_1^{\mathrm{new}},\dots,M_4^{\mathrm{new}})=(M_2,M_2^{-1}M_1M_2,M_3,M_4)$. We define $\mathrm{Braid}_{2,3}$ in the same way. As for $\mathrm{Braid}_{3,4}$, the conjugacy classes of M_3 and M_4 differ by a minus sign, so we compose $\mathrm{Braid}_{3,4}$ with multiplying M_3 and M_4 by -1. (Alternatively, one could only use *pure braids*, which map to the trivial permutation of $\{1,2,3,4\}$, such as $\mathrm{Braid}_{3,4}^2$.)

To obtain independence of basis changes, let $\sim$ be the equivalence relation on $\mathrm{GL}_4(\mathbb{C})^4$ where $(M_1,\dots,M_4)\sim(\tilde M_1,\dots,\tilde M_4)$ if they are simultaneously conjugated (that is, $\tilde M_i=P^{-1}M_iP$ for some P and all i). We denote the equivalence class of $(M_1,\dots,M_4)$ as

$$[M_1,\dots,M_4]_\sim\in\mathrm{GL}_4(\mathbb{C})^4/\!\sim.$$

The braid group $\langle\mathrm{Braid}_{1,2},\mathrm{Braid}_{2,3},\mathrm{Braid}_{3,4}\rangle$ acts on $\mathrm{GL}_4(\mathbb{C})^4/\!\sim$ well. We denote the orbit of the class $[M_1,\dots,M_4]_\sim$ as $O([M_1,\dots,M_4]_\sim)\subset\mathrm{GL}_4(\mathbb{C})^4/\!\sim$.

Like L^{PF}, the monodromy $(A_1,\dots,A_4)$ of L_+^{PF} is locally constant, and hence, changing the value its parameter t causes braid actions. The braid orbit contains all monodromies that can be obtained from L_+^{PF} for non-degenerate values of t. The simplest monodromy we found in the braid orbit was

$$\left[\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\\sqrt{2}-2&1\end{pmatrix},\begin{pmatrix}1&0\\-\sqrt{2}-2&1\end{pmatrix},\begin{pmatrix}1&-1\\4&-3\end{pmatrix}\right]_\sim.\tag{4.6}$$

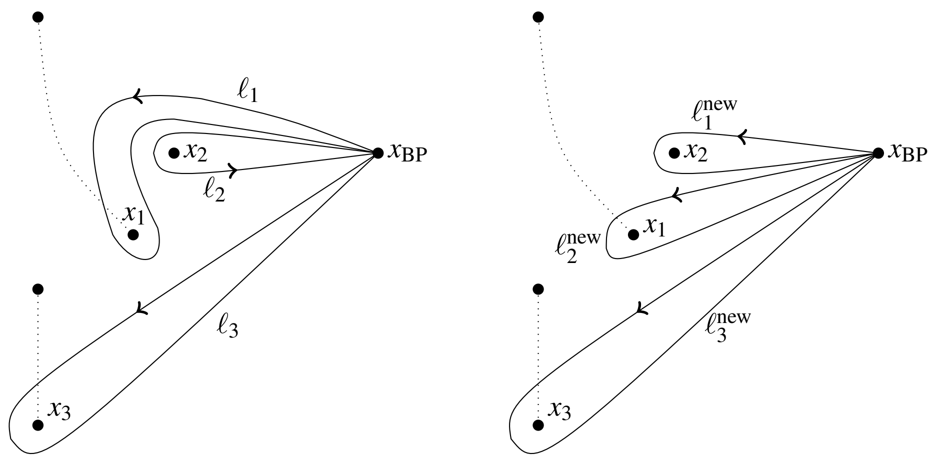

Figure 4.8: Positioning after a u-move

The following can be shown in a way similar to Section 5 in [vHK19].

Remark 4.1. Let L be a second-order regular singular differential equation, with four true singularities, and an arbitrary number of apparent singularities. If the monodromy matrices of L are in the above mentioned braid orbit, then L can be solved in terms of solutions of L_+^{PF} for some t. In particular, any such L must be arithmetic and have a solution that can be written in terms of a hyperelliptic integral.

Remark 4.2. Let ζ be a root of unity and let $\alpha = \zeta + \zeta^{-1} \in \mathbb{R}$. Let O_ζ be the braid orbit of the four matrices in $\mathrm{GL}_2(\mathbb{Z}[\alpha])$ obtained by replacing $\sqrt{2}$ in (4.6) with α.

By the Riemann–Hilbert correspondence there should exist a family of second-order differential equations with monodromy in O_ζ. Three singularities x_1, x_2, x_3 can be moved to $0, 1, \infty$ with Möbius transformations, leaving one singularity x_4 to move freely. The braid orbit O_ζ is *finite*, so we expect this family to be definable over a finite extension of $\mathbb{Q}(x_4)$. In other words, we expect an algebraically defined one-dimensional family of differential equations for O_ζ, just like L_+^{PF} for O_{ζ_8}.

Remark 4.3. As pointed out by the anonymous referee, certain aspects of our results in a different context appear in the literature. Recall that the Painlevé VI equation P_{VI} characterizes the isomonodromic deformation equation of second-order Fuchsian differential equations with *four* regular singular points and an apparent singular point on the Riemann sphere. If the monodromy tuple has a finite orbit, then the corresponding Painlevé VI equation has algebraic solutions; all such tuples are classified in [LT14]. The differential equation $L_\pm^{\mathrm{PF}}$ corresponds to the so-called Picard case $P_{\mathrm{VI}}(0,0,0,1)$, see [Maz01]; an Okamoto transformation (see, for example, [CL07, Appendix B]) brings the latter to the equation $P_{\mathrm{VI}}(1/2,1/2,1/2,1/2)$ that corresponds to a monodromy tuple with four reflections and the same action of the braid group [DR07, Theorem 2.2 and Example 6.5]. These four reflections generate an orthogonal subgroup of $\mathrm{GO}_2(\mathbb{C})$, as taking the second symmetric power yields a fixed space. This group is finite if and only if the braid group orbit is finite. Here the generated group is the dihedral group D_8 (of size 16) with braid orbit length 8. The Okamoto transformation above can be seen as a *middle convolution* with $x^{1/2}$ (compare with [DR07, Section 6]). Applying this middle convolution MC_{-1} to a symplectic monodromy tuple yields an orthogonal one; this explains why the Hadamard product of the periods with $(1-4x)^{1/2}$ is an algebraic function, which in turn, by a known result of Klein, is a pullback of a hypergeometric function. Furthermore, the middle convolution is invertible and can be written explicitly for the monodromy group generators. This gives one a sanity check that the generators of the monodromy tuple for $L_{\pm}^{\mathrm{PF}}$ are as claimed (up to braiding).

5 Around Dwork's conjecture

5.1 Dwork's conjecture

A conjecture of Dwork [Dwo90, p. 784] from 1990 suggests that *globally nilpotent* (over a number field) second-order differential equations are ${}_2F_1$-solvable; see [Dwo90, Section 7] for a precise statement. In this form, the conjecture was shown to be false by Krammer [Kra96] and later by Dettweiler, Reiter [DR10]. Bouw and Möller [BM10] (see also [MZ16]) gave examples of *globally bounded* but not ${}_2F_1$-solvable equations over the orders of some real quadratic extensions of $\mathbb{Q}$. Solving numerous second-order differential equations obtained from the OEIS [ST] suggested that this refined version of Dwork's conjecture might still be true over $\mathbb{Q}$ — this was explicitly stated as Conjecture 1 in [vHK19]. However, Corollary 4.1 combined with Section 4.2 strongly indicate that this is not the case.

All examples prior to our work have been Heun-solvable, that is, they are algebraic pullbacks of suitable Heun differential equations (with exactly four singularities). Our equation L_+^x has an additional apparent singularity, so it is not of Heun type. To give a more detailed comparison, we took an example from [MZ16], wrote it as a pullback of the Heun equation:

$$L_{\mathrm{Heun}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{768t^2 + 2(638-217\sqrt{17})t - 895 + 217\sqrt{17}}{t(t-1)(256t+895-217\sqrt{17})}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{60(4t+3-\sqrt{17})}{t(t-1)(256t+895-217\sqrt{17})}$$

and computed the monodromy of the latter. Up to equivalence, it is

$$\left[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -\frac{7}{2}-\frac{\sqrt{17}}{2} & 1\end{pmatrix}, \begin{pmatrix}-\frac{3}{2}-\frac{\sqrt{17}}{2} & \frac{9}{8}+\frac{\sqrt{17}}{8}\\ -\frac{13}{2}-\frac{3\sqrt{17}}{2} & \frac{7}{2}+\frac{\sqrt{17}}{2}\end{pmatrix}\right]_{\sim}$$

(the omitted 4th matrix is the inverse product of the other matrices). The braid orbit of this monodromy is not finite. That distinguishes it from our example L_+^x, which (recall Remark 4.2) is an example of the following type of family of differential equations: four non-apparent singularities that are moved non-trivially by an algebraic parameter t, where the monodromy is *locally constant* when we vary t (avoiding degenerate t-values where singularities collide). Any member of this type of family (obtained by substituting a non-degenerate value for t) will necessarily have a monodromy with a finite braid orbit. Thus L_{Heun}, with its infinite braid orbit, does not occur as a member *in this type of family*. Several features — defined over $\mathbb{Q}$, and an additional parameter — appear to be novel in our (expected to be a) counterexample to Dwork's conjecture for the globally bounded case.

Having discussed differences, similarities are the subject of Section 5.2 below. One starts with a Hilbert modular surface and a family of hyperelliptic curves on it.

5.2 Teichmüller curves on Hilbert modular surfaces

For an order $\mathcal{O}_D$ in a real quadric field of discriminant D, where $D \equiv 0$ or $1 \bmod 4$ is not a square, one forms the Hilbert modular surface

$$X_D := \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_D).$$

It can be interpreted as the moduli space of special principally polarised abelian surfaces, and the image of the corresponding map $X_D \to \mathcal{A}_2$ is the Humbert surface H_D. McMullen has shown (see [McM23] for an overview) that on each X_D there is one (primitive) Teichmüller curve, unless $D \equiv 1 \bmod 8$ in which case there are two such curves. It is known that the Picard–Fuchs equation for the corresponding family of $g = 2$ Veech curves over such a Teichmüller curve decomposes, one factor being the uniformising differential equation for the (open) Teichmüller curve. A salient feature of these Teichmüller curves is that the eigenform has a double zero on the hyperelliptic curve. The monodromy groups are discrete in $\mathrm{SL}_2(\mathbb{R})$, but not commensurable to the modular group $\mathrm{SL}_2(\mathbb{Z})$. The example of Bouw and Möller corresponds to the case $D = 17$ and their operator is defined over $\mathbb{Q}(\sqrt{17})$.

In our situation, we are dealing with a two-parameter family of hyperelliptic curves with real multiplication by $\sqrt{2}$. This means that our family is pulled back from the Hilbert modular surface X_8. For a general member $C^{u,x}$ of our family, the differential does not have a double zero. This shows that the phenomena described by Bouw and Möller are not due to the Teichmüller property, but seem to stem from the real multiplication itself and extend to the whole surface X_8. We now describe this double zero scenario for X_8. Let ω be the differential in equation (2.13), which is then of the form

$$\frac{\sqrt{-2}}{\pi}\int \omega$$

This ω is a differential of the first kind on our hyperelliptic curve, defined by the equation $Y^2 = H(v)$, where $H(v)$ denotes the sextic in v (depending also on t and x) from the square root of the denominator. In general, such a differential has $2g-2 = 2$ zeros, and they appear at the two points where the numerator of the differential ω vanishes: that is, when $v = 1-t$, hence $Y = \pm\sqrt{H(1-t)}$. By the theory of McMullen (see [McM23] for an overview) the case, where these two points coalesce, characterizes the Teichmüller condition inside the Humbert (or Hilbert) modular surface. It is clear that these two points can only coalesce when they become a root of the defining sextic. Therefore we are left to study the curve in the (t,x)-space defined by the condition $H(1-t) = 0$, which after the expansion reads $(t-1)(1-2t-128t(1-t)^3x)$. Ignoring the trivial double root originating from $t = 1$ we take a closer look at the one coming from

$$x = \frac{1-2t}{128t(1-t)^3}.$$

Substituting this into the integral (2.13) gives

$$I = \frac{2}{\pi}\int \frac{(t-1)^{3/2}(v+t-1)\,\mathrm{d}v}{\sqrt{(v+t)(tv+t^2-1)(v-t+1)(v+t-1)(v^2(1-2t)+2t^3-3t^2+1)}}.$$

This integral satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}I + \frac{2(8t^4-16t^3+6t-1)}{(t-1)(4t^2-1)(2t^2-1)}\frac{\mathrm{d}}{\mathrm{d}t}I + \frac{2t^4-4t^3+15t^2-6t-1}{(t-1)^2(4t^2-1)(2t^2-1)}I = 0.$$

Solving that gives

$$\frac{1-t}{t^{3/2}}\,{}_2F_1\left(\begin{matrix}\frac{3}{8},\frac{3}{8}\\1\end{matrix}\,\middle|\,\frac{4t^2-1}{4t^4}\right)$$

which equals I if the integral is taken from $v=-t$ to $v=t^{-1}-t$ and t is sufficiently large. The monodromy group of this ${}_2F_1$ is the (non-arithmetic!) triangle group $(4,\infty,\infty)$. Thus, our curve maps to the Teichmüller curve W_8.

Remark 5.1 (The case X_5)**.** The relation between X_8 and L_+^x motivated us to compute a similar family of differential equations for the surface X_5. After some transformations to reduce its size we found

$$\begin{aligned}L_5=\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{2x^3+(u^2+6u+1)x^2-16u^2}{x(x^3+(u^2+6u+1)x^2+8u(u+1)x+16u^2)}\frac{\mathrm{d}}{\mathrm{d}x}\\+\frac{(25x^2-(u+3)^2x-40u(u+3))}{100x(x^3+(u^2+6u+1)x^2+8u(u+1)x+16u^2)}.\end{aligned}$$

The monodromy of L_5, for any non-degenerate value of u, is in $O_{\zeta_5}=O_{\zeta_{10}}$ from Remark 4.2. A claim like Remark 4.1 applies here as well, every equation whose monodromy is in O_{ζ_5} can be solved in terms of L_5 and hence in terms of hyperelliptic integrals. Taking a local solution of L_5 at $x=\infty$, multiplying the local parameter by 5 and replacing u with $5u-3$, produces a power series whose coefficients in $\mathbb{Z}[u]$ are divisible by $\binom{2n}{n}$ and satisfy the recursion

$$\begin{aligned}&(n+1)^2a_{n+1}+5\big((5n+3)(5n+2)u^2-2(2n+1)^2\big)a_n\\&\quad+100(5u-3)\big(2(5u-2)n^2-3u+1\big)a_{n-1}+500(5u-3)^2(2n+1)(2n-3)a_{n-2}=0\end{aligned}$$

for $n=0,1,2,\dots$, with the convention $a_0=1$ and $a_n=0$ for $n<0$.

6 Cubes of Legendre polynomials

We now briefly turn our attention to the generating function (1.5).

Theorem 6.1. *Let* $p=1-xy^3+x^2(3y^2-2)/4$ *and*

$$H=\frac{1}{\sqrt{p}}\cdot{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{3}{4}\\1\end{matrix}\,\middle|\,\frac{(y^2-1)^3(x^2-\frac{1}{4}x^4)}{p^2}\right).$$

Then

$$F_{P^3}(y,z)=\frac{1}{\sqrt{1+z^2}}\left(H\star\frac{1}{\sqrt{1-4x}}\right)\bigg|_{x=z/(1+z^2)}$$

where the Hadamard product $\star$ is with respect to the internal variable x.

Again it is relatively easy to verify this formula, and the proof below mainly outlines how it was found.

Proof. First observe $z \mapsto 1/z$ symmetry in the singularities. That suggests that we can write the equation in terms of $x = z/(1+z^2)$, a generator of the subfield of $\mathbb{Q}(z)$, which is fixed under the symmetry. Indeed, the corresponding change of variables produces the equation that is nearly rational in x (it is rational after a simple multiplication of solutions by an algebraic factor). Then unexpectedly, like in subsection 2.3, we observe that the new equation has the 'central binomial twist' property: the coefficients of its analytic solution at the origin can be divided by $\binom{2n}{n}$ leading to a series which is still globally bounded. More surprisingly, the differential equation for the resulting series has order 2! A closed form (hypergeometric) solution can then be found quickly. □

The fourth-order differential operator L_{P^3} for (1.5) with respect to the variable z decomposes as a symmetric product of second-order equations just as in Section 2. But the coefficients are in a genus 1 function field. The question remains if these second-order equations can also be solved in terms of algebraic integrals. We hoped to find a connection to some O_ζ from Remark 4.2, but those have 4 logarithmic singularities, a feature we did not encounter in equations we were able to obtain from L_{P^3}.

7 Conclusion and future work

A task of recognising the innocent-looking generating function (1.4) of the squares of Legendre polynomials in a closed form has turned out to produce families of hyperelliptic integrals that satisfy second-order Picard–Fuchs differential equations, with the latter having remarkable features. Next one can investigate other Humbert surfaces corresponding to different orders, and we did for X_5 in Remark 5.1. The corresponding second-order equations are solved by hyperelliptic integrals with the underlying Jacobians admitting RM by $\mathbb{Z}[(1+\sqrt{5})/2]$. We plan to do this for more Humbert surfaces elsewhere; the underlying geometric structures for small orders are already classified in [EK14, KM16].

After analyzing the relation between X_8 and O_{ζ_8} from Remark 4.2, and X_5 and O_{ζ_5} in Remark 5.1, the first author constructed a program (available at [vH23]) that takes as input a positive integer N and a non-cuspidal point P on $X_1(N)$. It computes a second-order differential operator $L_{N,P}$, and a hyperelliptic integral that satisfies $L_{N,P}$, both defined over $\mathbb{Q}(P)$. To our surprise, not only can hyperelliptic integrals satisfy second-order equations, but also, we now have a program that can produce examples of arbitrarily high genus. This $L_{N,P}$ conjecturally has monodromy in O_{ζ_N}, implying that the algorithm produces families of hyperelliptic curves with real multiplication by $\mathbb{Z}[\zeta_N + \zeta_N^{-1}]$, as in Mestre [Mes91].

Interestingly, the discussed surface X_8 (and it only!) has received a differential treatment 35 years ago in [SY88]: as an illustration of their general construction, Sasaki and Yoshida explicitly gave a rank 4 system of partial differential equations for the Hilbert surface X_8. We expect that system to be related to the one from Section 2. To test this we checked that it has a similar decomposition to rank 2, which it does.

Our proof of Theorem 2.1 originates the following expectation of an arithmetic flavour, which may be regarded as a potential correction of Dwork's ex-conjecture. Assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is in $1 + x\mathbb{Z}[[x]]$ and that its twist $\sum_{n=0}^{\infty} \binom{2n}{n} a_n x^n$ by the central binomial coefficients satisfies a second-order linear differential equation with regular singularities. Must $f(x)$ be algebraic?

Our proofs of Theorems 2.1 and 6.1 also demonstrate that closed forms of arithmetic (globally bounded) series may be obtained from 'untwisting' the coefficients by the central binomial coefficients $\binom{2n}{n}$ (or more generally by $(\alpha)_n/n!$ for some $\alpha \in \mathbb{Q}\setminus\mathbb{Z}$). Such untwisting may depend on a special choice of the variable, with respect to which the series are expanded. The choices required in our proofs were *ad hoc*, though hinted by the corresponding differential equations. We would therefore question whether there exists a *routine* strategy for performing the related change of variable directly from the equations, in a greater generality. Such a methodology, if it exists, has potential to shed more light on a famous conjecture of Christol [Chr15] about globally bounded solutions of Picard–Fuchs differential equations.

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AUTHORS

Mark H. F. van Hoeij

Florida State University
1017 Academic Way
Tallahassee, FL 32304
United States

hoeij@math.fsu.edu
http://www.math.fsu.edu/~hoeij/

Duco van Straten

Fachbereich 08
Johannes Gutenberg University
AG Algebraische Geometrie
55099 Mainz
Germany

straten@mathematik.uni-mainz.de

Wadim Zudilin

Institute for Mathematics, Astrophysics and Particle Physics
Radboud University
PO Box 9010
6500 GL Nijmegen
Netherlands

w.zudilin@math.ru.nl
https://www.math.ru.nl/~wzudilin/