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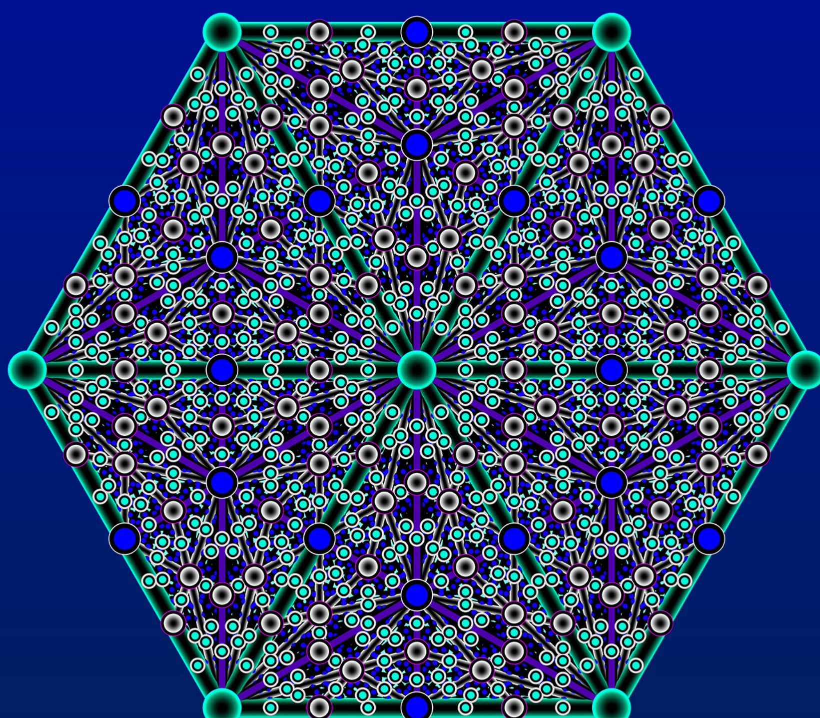

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Hilbert Coefficients of Quadratic Algebras

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Abstract:
If $R = k[x_1,\dots,x_n]/I$ is a graded artinian algebra, then the length of $k[x_1,\dots,x_n]/I^s$ becomes a polynomial in s of degree n for large s. If we write this polynomial in the form $\sum_{i=0}^{n}(-1)^i e_i\binom{s+n-i-1}{n-i}$, then the e_i's are called Hilbert coefficients of I. We will study lengths and Hilbert coefficients of some classes of quadratic algebras.

Key words and phrases:
Hilbert coefficient; Quadratic algebra; Hilbert series

1 Introduction

Let $S_n = k[x_1,\dots,x_n]$, k a field, and let I be a graded ideal in S_n, such that S_n/I is Artinian. Let $S_n/I^s(t) = \sum_{i\geq 0} h_i t^i$, where $h_i = \dim_k(S_n/I^s)_i$, be the Hilbert series of S_n/I^s (which is a polynomial since S_n/I^s is artinian) and $l(S_n/I^s) = \sum_{i\geq 0} h_i$ the length of S_n/I^s. Samuel has shown [Sam51] that the length of S_n/I^s is a polynomial in s of degree n for all large values of s. If we write this polynomial in the form

$$e_0(I)\binom{s+n-1}{n}+\cdots+(-1)^i e_i(I)\binom{s+n-i-1}{n-i}+\cdots+(-1)^n e_n(I),$$

then the e_i's are called the Hilbert coefficients of I, and in particular $e_0(I)$ is called the multiplicity of I. In fact, Samuel and others consider the more general case when I is primary for the maximal ideal in a Noetherian local ring, but we stick to the graded case. The first paper on Hilbert coefficients that we know of is by Northcott [Nor60]. There it is shown that if I is an m-primary ideal in a CM local ring (Q,m), then $e_1(I)\geq 0$ with equality if and only if I is generated by a system of parameters. Later, Narita [Nar63] showed that with the same conditions $e_2(I)\geq 0$ and gave conditions for equality. He also showed that $e_3(I)$ could be negative. Later, most articles on Hilbert coefficients have dealt with connection to depth of the associated graded ring, and with different generalizations of the concept of Hilbert coefficients.

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We will study the Hilbert series $Hilb_R(t) = \sum_{i\geq 0} \dim_k R_i t^i$ and the Hilbert coefficients for artinian quadratic algebras, i.e. for rings $R = S_n/I^s$, I generated in degree 2. The Hilbert series gives the length as $l(R) = Hilb_R(1)$. Based on extensive calculations we give the following conjecture.

Conjecture 1.1. *If S_n/I is a quadratic algebra, then $e_i(I) \geq 0$ for all i.*

We will start, in Section 2, with ideals generated by general elements. If I is generated by n general elements, the smallest possible number, then S/I is a complete intersection, i.e. I is generated by a regular sequence. In that case the Hilbert series of S_n/I^s is more or less known. We try to give a bit more explicit expressions for the Hilbert series for quadratic algebras. If I is generated by more than n general elements, very little is known, and we give results and conjectures for the Hilbert series of S_n/I^s, and thus for the Hilbert coefficients, in some small cases. Then, in Section 3, we turn to the opposite, namely we consider (quadratic) monomial ideals. We start with the extreme case $I = (x_1,\dots,x_n)^2$ in S_n. Here the determination of the Hilbert series is trivial, but we have a nice conjecture for the Hilbert coefficients of I^s, checked for $n \leq 11$. Finally we consider all artinian rings with quadratic monomial relations in three and four variables.

2 Ideals generated by general (quadratic) elements

The following is well known [Fil67, Theorem1.2], [Nag56, Prop. 6, p. 208].

Theorem 2.1. *Let $f_1,\dots,f_n$ be a graded regular sequence in $S_n = k[x_1,\dots,x_n]$, k a field, $\deg(f_i) = d_i$. If $Q = (f_1,\dots,f_n)$, then*

$$l(S_n/Q^s) = l(S_n/Q)\binom{s+n-1}{n}.$$

Corollary 2.2. *We have $e_0(S_n/Q^s) = \prod_{i=1}^n d_i$ and $e_i(S_n/Q) = 0$ if $i > 0$.*

Proof. This follows from the definition of the e_i's and from $l(S_n/Q) = \prod_{i=1}^n d_i$. $\square$

The graded Betti numbers for S_n/Q^s are described in [GvT05]. As a corollary they got an expression for the Hilbert series. (A similar theorem for the Hilbert series exists in [Fil67, Theorem 4.1].) We will give a formula for the Hilbert series of S_n/Q^s which is a bit more explicit in the case when Q is generated in degree 2. We now describe the result on the Hilbert series in [GvT05].

Let $L_{n,s} = \{(a_1,\dots,a_n) \in \mathbb{N}^n | \sum_{i=1}^n a_i \leq s-1\}$.

Theorem 2.3. *[GvT05, Corollary 2.3]*

$$H_{S_n/Q^s}(i) = \sum_{(a_1,\dots,a_n)\in L_{n,s}} H_{S_n/Q}\Big(i - \sum_{i=1}^n a_i d_i\Big)$$

where H_R is the Hilbert function of R.

We now specialize to $d_i = 2$ for all i. Then the theorem gives that $H_{S_n/Q^s}(i) = \sum_{(a_1,\dots,a_n)\in L_{n,s}} H_{S_n/Q}(i - 2\sum_{i=1}^n a_i)$. We now suppose that Q is generated by a regular sequence of degree 2 and length n in $S_n = k[x_1,\dots,x_n]$.

Corollary 2.4. *The highest degree where $H_{S_n/Q^s}(i)$ is non-zero is $i = 2s + n - 2$.*

Proof. $S_n/Q(t) = (1+t)^n$, so it exists in degrees $\leq n$. Then the top degree of $S_n/Q^s(t)$ is achieved at $\max\{i; i - 2\sum_{i=1}^{n} a_i = n\}$, i.e. when $i = n + 2(s-1)$. ☐

Lemma 2.5. *There are $\binom{n+j-1}{n-1}$ instances for $\sum a_i = j$.*

Proof. This is the number of monomials of degree j in n variables. ☐

Corollary 2.6. *The Hilbert series of S_n/Q^s, Q generated by a regular sequence in degree 2 of length n is*

$$\sum_{i=0}^{2s-1} \binom{i+n-1}{n-1} t^i + \sum_{i=2s}^{2s+n-2} \sum_{j \leq s-1} \binom{n+j-1}{n-1} \binom{n}{i-2j} t^i.$$

Proof. The expressions in both the first and second sum follow from Lemma 2.5. ☐

Corollary 2.7. $\binom{n+2s}{n} + \sum_{i=2s}^{2s+n-2} \sum_{j \leq s-1} \binom{n+j-1}{n-1} \binom{n}{i-2j} = 2^n \binom{s+n-1}{n}$.

Proof. $\sum_{i=0}^{2s-1} \binom{i+n-1}{n-1} = \binom{n+2s}{n}$ since the number of monomials of degree *at most* $2s-1$ in n variables equals the number of monomials of *exactly* degree $2s-1$ in $n+1$ variables. ☐

To get more explicit expressions for the Hilbert series, we further restrict the number of variables.

Example 2.1. *Let $n = 2$. Then the Hilbert series of S_2/Q^s is*

$$\sum_{i=0}^{2s-1} (s+1)t^i + st^{2s}.$$

Let $n = 3$. Then the Hilbert series of S_3/Q^s is

$$\sum_{i=0}^{2s-1} \binom{i+2}{2} t^i + 3\binom{s+1}{2} t^{2s} + \binom{s+1}{2} t^{2s+1}.$$

Let $n = 4$. Then the Hilbert series of S_4/Q^s is

$$\sum_{i=0}^{2s-1} \binom{i+3}{3} t^i + (\binom{s+1}{3} + 6\binom{s+2}{3}) t^{2s} + 4\binom{s+2}{3} t^{2s+1} + \binom{s+2}{3} t^{2s+2}$$

Let $n = 5$. Then the Hilbert series of S_5/Q^s is

$$\sum_{i=0}^{2s-1} \binom{i+4}{4} t^i + (5\binom{s+2}{4} + 10\binom{s+3}{4}) t^{2s} + (\binom{s+2}{4} + 10\binom{s+3}{4}) t^{2s+1} + 5\binom{s+3}{4} t^{2s+2} + \binom{s+3}{4} t^{2s+3}.$$

Calculating the length we get:

Corollary 2.8.

$$\binom{2s+1}{2} + s = 4\binom{s+1}{2}.$$

$$\binom{2s+2}{3} + 4\binom{s+1}{2} = 8\binom{s+2}{3},$$

$$\binom{2s+3}{4} + \binom{s+1}{3} + 11\binom{s+2}{3} = 16\binom{s+3}{4},$$

$$\binom{2s+4}{5} + 6\binom{s+2}{4} + 26\binom{s+3}{4} = 32\binom{s+4}{5}.$$

We continue to consider ideals generated in degree 2. Complete intersections are the most general ideals with n generators in S_n. We will now look at general ideals I with more than n generators and study S_n/I^s. This is much harder, since not even the Hilbert series for $s=1$ is known, unless $n \leq 3$ [Ani86], or we have $n+1$ generators of the ideal [Sta78]. In both these cases the Hilbert series of S_n/I, I generated by r generic forms of degree 2, is $(1+t)^n(1-t^2)^{r-n}$ truncated just before the first nonpositive coefficient. Let $R_{n,r,s}$ be S_n/I^s, I generated by r general elements of degree two. The Hilbert series of $R_{n,r,s}$ is known only for $n=2$ and for $(r,n)=(4,3)$ as far as we know. We have

Theorem 2.9. *[BFL18, Theorem 8] The Hilbert series of $R_{3,4,s} = k[x,y,z]/(f_1,f_2,f_3,f_4)^s$, f_i general forms of degree 2, is*

$$\sum_{i=0}^{2s-1} \binom{i+2}{2} t^i + (3s-1)t^{2s}.$$

Corollary 2.10. *The Hilbert coeficients of $R_{3,4,s} = k[x,y,z]/(f_1,f_2,f_3,f_4)^s$, f_i general forms of degree 2, are $(e_0,e_1,e_2,e_3) = (8,4,3,4)$.*

It is clear that the Hilbert series of $k[x_1,\ldots,x_n]/I^s$, I generated by r forms of degree 2, is smallest when the generators are generic. It is easy to find the smallest Hilbert series for $R_{3,5,s}$.

Theorem 2.11. *The Hilbert series of $R_{3,5,s} = k[x,y,z]/(f_1,f_2,f_3,f_4,f_5)^s$, f_i generic of degree 2, is*

$$\sum_{i=0}^{2s-1} \binom{2i+2}{2} t^i$$

if $s \geq 2$. Thus $(f_1,f_2,f_3,f_4,f_5)^s = (x,y,z)^{2s}$ if $s \geq 2$.

Proof. It suffices to find an ideal generated by five forms of degree 2 with the claimed Hilbert series. We choose $I = (x^2,y^2,z^2,xy+xz+yz,xz+2yz)$. A calculation shows that $I^s = (x,y,z)^{2s}$ for $s=3,4,5$. Any $n \geq 3$ can be written as $n = 3a+4b+5c$ so $I^n = (I^3)^a(I^4)^b(I^5)^c = m^{3a}m^{4b}m^{5c} = m^n$, where $m = (x,y,z)$. □

Question 2.1. *Let $\phi(n)$ be the smallest s such that $I^s_{r,n} = (x_1,\ldots,x_n)^{2s}$. Can one determine $\phi(n)$? Is $\phi(n) = 2n-1$?*

We have seen the $\phi(3)=5$, and we conjecture that $\phi(4)=7$, $\phi(5)=9$.

Conjecture 2.1.

- *The Hilbert series of $R_{4,5,s}$ is $\sum_{i=0}^{2s-1}\binom{i+3}{3}t^i + 10s^2 - 25s + 35$ if $s \geq 4$.*
  The Hilbert coefficients are $(e_0,e_1,e_2,e_3,e_4) = (16,12,21,35,35)$.
- *The Hilbert series of $R_{4,6,s}$ is $\sum_{i=0}^{2s-1}\binom{i+3}{3}t^i + 10s$ if $s \geq 4$.*
  The Hilbert coefficients are $(e_0,e_1,e_2,e_3) = (16,12,1,10)$.
- *The Hilbert series of $R_{4,7,s}$ is $\sum_{i=0}^{2s-1}\binom{i+3}{3}t^i$ if $s \geq 3$.*
  The Hilbert coefficients are $(e_0,e_1,e_2) = (16,12,1)$.
- *The Hilbert series of $R_{5,9,s}$ is $\sum_{i=0}^{2s-1}\binom{i+4}{4}t^i$ if $s \geq 4$.*
  The Hilbert coefficients are $(e_0,e_1,e_2) = (32,32,6)$.

3 Monomial ideals

The smallest ideal generated in degree 2 which gives an artinian ring is the complete intersection. Now we study the largest ideal generated in degree 2, namely $(x_1,\dots,x_n)^2$. If $I=(x_1,\dots,x_n)^2$ there is of course no problem to determine $l(S_n/I^s)$, it is $\binom{n+2s-1}{n}$. Here the problem is to determine the Hilbert coefficients. We have made extensive calculations, which lead to the following conjectures.

Conjecture 3.1. *If $S_n=k[x_1,\dots,x_n]$ and $I=(x_1,\dots,x_n)^2$, then*

- $e_i(S_n/I^s)\neq 0$ *exactly if* $i\leq\lfloor n/2\rfloor$.
- *We have* $e_j=\binom{n-j}{j}2^{n-2j}$ *if* $n\geq 2j$.

  thus
- $\binom{2s-1+n}{n}=\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j 2^{n-2j}\binom{n-j}{j}\binom{s+n-j}{n-j}$.

Example 3.1. *Let $s=5,n=4$. We have $l(S_4/(x_1,x_2,x_3,x_4)^{2\cdot 5})=\binom{13}{4}$, and $\binom{13}{4}=16\binom{8}{4}-4\cdot 3\binom{7}{3}+\binom{6}{2}$.*

We have checked the truth of Conjecture 3.1 for $n\leq 11$. It would be nice to have combinatorial proofs.

We continue to consider monomial ideals are generated in degree two. Let M be a set of squarefree monomials of degree two. We denote by $I_{n,M}$ the ideal in $S_n=k[x_1,\dots,x_n]$ which is generated by all squares together with those squarefree which are in M. If I is a monomial ideal generated in degree two in S_n containing all squares of the variables, we denote the interesting part of the Hilbert series of S_n/I^s, namely $\sum_{i\geq 2s}\dim_k S_n/I^s$, by $H_s(S_n/I^s)$.

We will determine $H_s(S_n/I^s_{n,M})$ and the Hilbert coefficients for all cases in three variables, and conjecture them for all cases in four variables. First, if $M=\emptyset$, we get the complete intersection $k[x_1,\dots,x_n](x_1^2,\dots,x_n^2)$, and for $M=\{x_ix_j;\ 1\leq i<j\leq n\}$ we get $k[x_1,\dots,x_n]/(x_1,\dots,x_n)^2$. These rings are already treated.

Now suppose we have three variables.

Theorem 3.1. *If $M=\{x_1x_2\}$, then $H_s(S_n/I^s_{n,M})=2\binom{s+1}{2}t^{2s}$ if $n\geq 2$. The Hilbert coefficients are $(e_0,e_1)=(8,6)$.*

If $M=\{x_1x_2,x_1x_3\}$, then $H_s(S_n/I^s_{n,M})=st^{2s}$ if $n\geq 2$. The Hilbert coefficients are $(e_0,e_1,e_2)=(8,4,1)$.

Proof. If $M=\{x_1x_2\}$, then the nonzero monomials in $S_3/(x_1^2,x_2^2,x_3^2,x_1x_2)^s$ of degree 2s are the generators of $(x_1x_3,x_2x_3)(x_1^2,x_2^2,x_3^2)^{s-1}$, and $S_3/(x_1^2,x_2^2,x_3^2,x_1x_2)^s$ is 0 in degree $2s+1$. If $M=\{x_1x_2,x_1x_3\}$, then the nonzero monomials in $S_3/(x_1^2,x_2^2,x_3^2,x_1x_2,x_1x_3)^s$ of degree 2s are the generators of $x_2x_3(x_2^2,x_3^2)^{s-1}$ and $S_3/(x_1^2,x_2^2,x_3^2,x_1x_2,x_1x_3)^s$ is 0 in degree $2s+1$. □

Now we turn to four variables.

Conjecture 3.2.

- *If* $M = \{x_1x_2\}$, *then* $H_s(S_n/I^s_{n,M}) = \frac{2s^3+5s^2+3s}{2}t^{2s} + 2\binom{s+2}{3}t^{2s+1}$.
  The Hilbert coefficients are $(e_0, e_1) = (16, 4)$.
- *If* $M = \{x_1x_2, x_1x_3\}$, *then* $H_s(S_n/I^s_{n,M}) = 4\binom{s+2}{3}t^{2s} + \binom{s+1}{2}t^{2s+1}$.
  The Hilbert coefficients are $(e_0, e_1, e_2) = (16, 16, 2)$.
- *If* $M = \{x_1x_2, x_3x_4\}$, *then* $H_s(S_n/I^s_{n,M}) = 4\binom{s+2}{3}t^{2s}$.
  The Hilbert coefficients are $(16, 16, 1)$.
- *If* $M = \{x_1x_2, x_1x_3, x_2x_3\}$, *then* $H_s(S_n/I^s_{n,M}) = \frac{s(s+1)(4s+5)}{6}t^{2s}$.
  The Hilbert coefficients are $(e_0, e_1) = (16, 8)$.
- *If* $M = \{x_1x_2, x_1x_3, x_1x_4\}$, *then* $H_s(S_n/I^s_{n,M}) = s(2s+1)t^{2s} + \binom{s+1}{2}t^{2s+1}$.
  The Hilbert coefficients are $(e_0, e_1, e_2) = (16, 12, 6)$.
- *If* $M = \{x_1x_2, x_2x_3, x_3x_4\}$, *then* $H_s(S_n/I^s_{n,M}) = s(2s+1)t^{2s}$.
  The Hilbert coefficients are $(16, 12, 5)$.
- *If* $M = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$, *then* $H_s(S_n/I^s_{n,M}) = 2st^{2s}$.
  The Hilbert coefficients are $(16, 12, 1, 2)$.
- *If* $M = \{x_1x_2, x_1x_3, x_1x_4, x_2x_3\}$, *then* $H_s(S_n/I^s_{n,M}) = 2\binom{s+1}{2}t^{2s}$.
  The Hilbert coefficients are $(16, 12, 3)$.
- *If* $M = \{x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4\}$, *then* $H_s(S_n/I^s_{n,M}) = st^{2s}$.
  The Hilbert coefficients are $(16, 12, 1, 1)$.

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